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# Conic approximation of planar curves

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**Abstract**

An upper bound of the Hausdorff distance between planar curve and conic section can be expressed by the maximum norm of error function from the conic section to the planar curve (Comput. Aided Geomet. Design, 14 (1997) 135–151). With respect to the maximum norm we characterize the necessary and sufficient condition for the conic section to be optimal approximation of the given planar curve. As an example, we approximate the cubic rational Bézier curves by conic sections using our characterization, and present the upper bound of the Hausdorff distance numerically. 

*Keywords*: Conic section; Optimization; Hausdorff distance; $G^1$ end-points interpolation; Maximum norm

## 1. Introduction

Conic spline or quadratic rational spline is a composite curve of conic sections or quadratic rational Bézier curves [7,9]. It is one of the most widely used curves in industry, e.g. to design the bodies of aircraft, to design the outlines of fonts [10,14] or to express circular arcs, spheres or tori [11,13,16–18]. Thus conic interpolation of the given planar curve and conic fitting of the planar data are frequently occurring problems in CAGD.

Any conic section can be written in the standard quadratic rational Bézier form

$$\mathbf{b}(t) = \frac{B_0(t)\mathbf{b}_0 + \mu B_1(t)\mathbf{b}_1 + B_2(t)\mathbf{b}_2}{B_0(t) + \mu B_1(t) + B_2(t)}$$

where $\mathbf{b}_i$, $i = 0, 1, 2$, are the control-points, the weight $\mu > 0$ associated with $B_1(t)$ is called the fullness factor of the conic section [1,7], and $B_0(t) = (1-t)^2$, $B_1(t) = 2t(1-t)$ and $B_2(t) = t^2$ are the quadratic Bernstein polynomials. Conic approximation including $G^1$ (tangent continuous) end-points interpolation is determined only by choosing the weight $\mu$ as shown in Fig. 1. Also, the $G^1$ end-point interpolating conic section of the given curve can be improved to $C^1$ end-points interpolation by change of weights using Möbius transformation [7].

In previous works [4,6,12,15] for conic spline approximation the weight $\mu$ is chosen so that the conic passes through a point or the parametric middle point of the given planar curve, or the conic spline satisfies a constrained curvature continuity. Most of them could not yield the optimal conic approximation having the minimal Hausdorff distance, since it is not easy to find the Hausdorff distance between the planar curve and the approximate conic section. Floater [8] presented a formula for the upper bound of the Hausdorff distance between planar curves and conic section curves. The formula is expressed in terms of the maximum norm of error function from the conic to the planar curve. In this paper, with respect to the maximum norm, we characterize the necessary and sufficient condition for the conic section to have the minimal maximum norm. Although our characterization does not yield the 'real' best conic approximation which is obtained by the minimization of the Hausdorff distance directly, the method looking for the best approximation cannot yield the explicit form of the error function so that its algorithm is more complicated than that of our method. As an application, we present the numerical results of approximation for the outline of the font '**r**' consisting of cubic rational Bézier curves using the composite of conic section curves.

In the following sections we characterize the necessary and sufficient condition for the conic section to be optimal approximation to the given planar curve with respect the maximum norm. We apply the characterization theorem to approximate the cubic rational spline curves by conic spline curves, and give the numerical results. We also summarize our work.

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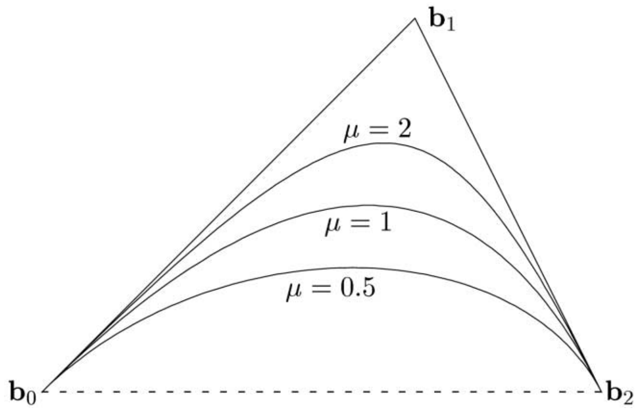


Fig. 1. The relation of the conic sections in form of standard quadratic rational Bézier curves having the control-points $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$, and the weights (fullness factor $\mu = 0.5$, 1 or 2).

## 2. Characterization of optimal conic approximation

In this section we present a method of approximation for the given planar curve using conic spline in $G^1$ end-points interpolation manner. We adopt the following procedures for the approximation scheme in this paper.

I. Input the planar curve.
II. Subdivide the curve until each subdivided segment can be contained in (the interior of) a triangle, say $\Delta\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$, and any conic section $\mathbf{b}(t)$ having the control-points $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$, is a $G^1$ end-points interpolation of the segment.
III. Repeat approximation and subdivision until the upper bound of the Hausdorff distance between the segment and the conic section $\mathbf{b}_\mu(t)$ with the weight $\mu$ obtained by approximation method is less than given tolerance.
IV. Merge all the approximate conic sections.

In step II or III, if a subdivision is needed, the curve may be subdivided at cusp, inflection point or largest-distance point from the line $\mathbf{b}_0\mathbf{b}_1$. Step III is the main part of approximation for the planar curve using the conic spline. Thus we may assume that the given curve $\mathbf{p}(s)$, $s \in [a, b]$, lies in $\Delta\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$, and any conic section $\mathbf{b}(t)$ having the control-points $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$ and the weight $\mu$ is a $G^1$ end-points interpolation of $\mathbf{p}(s)$. Thus the conic approximation is determined by only the weight, and our strategy is to find the weight $\mu$ such that $\mathbf{b}_\mu(t)$ is the optimal approximation of the given planar curve $\mathbf{p}(s)$ with respect to the maximum norm presented by Floater [8].

Any point $\mathbf{x}$ in the plane can be written uniquely in terms of barycentric coordinates $\tau_0$, $\tau_1$, $\tau_2$, where $\tau_0 + \tau_1 + \tau_2 = 1$, with respect to $\Delta\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$: $\mathbf{x} = \tau_0\mathbf{b}_0 + \tau_1\mathbf{b}_1 + \tau_2\mathbf{b}_2$ provided $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$ are not collinear. Thus any function defined on $\Delta\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$ can be expressed as a function of $\tau_0$, $\tau_1$, $\tau_2$. A class of functions $f\mu : \Delta\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2 \longrightarrow \mathbf{R}$ is defined by [7]

$$f_\mu(\mathbf{x}) = 4\mu^2\tau_0\tau_2 - \tau_1^2$$

Using the function Floater [8] presents a formula for an upper bound of the Hausdorff distance between the conic sections and the curves contained in $\Delta\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$.

**Theorem 1.** *For any continuous curve $\mathbf{p}(s)$, $s \in [a, b]$, contained in $\Delta\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$, the Hausdorff distance between $\mathbf{p}(s)$ and the conic section $\mathbf{b}_\mu(t)$ is bounded by*

$$d_{\mathrm{H}}(\mathbf{b}_\mu, \mathbf{p}) \le \frac{1}{4} \max\left\{1, \frac{1}{\mu^2}\right\} \max_{s\in[a,b]} |f_\mu(\mathbf{p}(s))|\, |\mathbf{b}_0 + \mathbf{b}_2 - 2\mathbf{b}_1| \tag{1}$$

**Proof.** See Floater [8]. □

We define the error function $\psi_\mu(s)$ on $[a, b]$ by

$$\psi_\mu(s) := \frac{1}{4} \max\left\{1, \frac{1}{\mu^2}\right\} f_\mu(\mathbf{p}(s)) |\mathbf{b}_0 + \mathbf{b}_2 - 2\mathbf{b}_1|$$

The maximum norm of $\psi_\mu(s)$ is the upper bound in Theorem 1. Note that the upper bound (1) overestimates the Hausdorff distance as the weight $\mu$ or the ratio of the lengths of two legs $\mathbf{b}_0\mathbf{b}_1$ and $\mathbf{b}_1\mathbf{b}_2$ differ from one.

As an example of curves contained in the triangle $\Delta\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$, we give the quartic polynomial curve $\mathbf{p}(s)$ parameterized by

$$\mathbf{p}(s) = (1 - 8s + 26s^2 - 32s^3 + 13s^4)\mathbf{b}_0 - 4(-2s + 9s^2 - 14s^3 + 7s^4)\mathbf{b}_1 + (10s^2 - 24s^3 + 15s^4)\mathbf{b}_2$$

for $s \in [0, 1]$, and the conic sections $\mathbf{b}_\mu(t)$ as shown in Fig. 2a. Assuming $|\mathbf{b}_0 + \mathbf{b}_2 - 2\mathbf{b}_1| = 2$, the error functions $\psi_\mu(s)$, $\mu = \sqrt{2}, 1, 1/\sqrt{2}$, are plotted in Fig. 2b, whose uniform norms

$$\|\psi_\mu(s)\|_{L^\infty} := \max_{s\in[0,1]} |\psi_\mu(s)|$$

equal 0.517, 0.243 and 0.246, respectively.

With respect to the maximum norm (1), we present the necessary and sufficient condition for the conic section to be optimal approximation of the planar curve using the nonlinear approximation theory, that is to say, the error function of the optimal conic approximation must be equioscculating as many times as possible. The properties of the class of error functions $\psi_\mu(s)$ in the following lemmas are needed to characterize the optimal conic approximation for the given planar curve with respect to the maximum norm (1).

**Lemma 1.** *Any distinct two error functions, $\psi_{\mu_1}(s)$ and $\psi_{\mu_2}(s)$, for $\mu_1 \ne \mu_2$, have not any intersection points in the open interval (a,b), i.e., $\psi_{\mu_1}(s) \ne \psi_{\mu_2}(s)$ for all $s \in (a, b)$ whenever $\mu_1 \ne \mu_2$.*

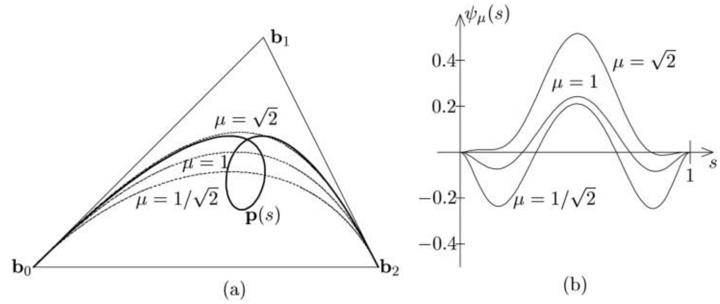


Fig. 2. (a) The quartic polynomial curve $\mathbf{p}(s)$, $s \in [0, 1]$, contained in the triangle $\Delta\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$, plotted by solid lines, and the conic sections, $\mu = \sqrt{2}, 1, 1/\sqrt{2}$, by dashed lines. (b) The error function $\psi_\mu(s)$ from $\mathbf{p}(s)$ to the conic sections $\mathbf{b}_\mu(t)$, $\mu = \sqrt{2}, 1, 1/\sqrt{2}$.

**Proof.** For each fixed $s \in (a,b)$, the point $\mathbf{p}(s)$ has the fixed barycentric coordinate $(\tau_0, \tau_1, \tau_2)$. Thus

$$\max\left\{1, \frac{1}{\mu^2}\right\} f_\mu(\mathbf{p}(s)) = \begin{cases} 4\mu^2\tau_0\tau_2 - \tau_1^2 & \text{for } \mu \geq 1 \\ 4\tau_0\tau_2 - \tau_1^2/\mu^2 & \text{for } \mu < 1 \end{cases}$$

is strictly increasing with respect to $\mu$ because $\tau_i > 0$, $i = 0, 1, 2$, so is $\psi_\mu(s)$. Hence $\psi_{\mu_1}(s) \neq \psi_{\mu_2}(s)$ for all $s \in (a,b)$ if $\mu_1 \neq \mu_2$. □

**Lemma 2.** *For each $\mu$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|\psi_\mu(s) - \psi_{\mu'}(s)\|_{L^\infty} < \varepsilon$$

if $|\mu - \mu'| < \delta$.

**Proof.** For each $\mu$, we have the following inequality:

$$\frac{4|\psi_\mu(s) - \psi_{\mu'}(s)|}{|\mathbf{b}_0 + \mathbf{b}_2 - 2\mathbf{b}_1|} =$$

$$\left| f_\mu(\mathbf{p}(s)) \max\left\{1, \frac{1}{\mu^2}\right\} - f_{\mu'}(\mathbf{p}(s)) \max\left\{1, \frac{1}{\mu'^2}\right\} \right|$$

$$= \Bigg| 4\tau_0\tau_2(\max\{\mu^2, 1\} - \max\{\mu'^2, 1\})$$

$$- \tau_1^2\left(\max\left\{1, \frac{1}{\mu^2}\right\} - \max\left\{1, \frac{1}{\mu'^2}\right\}\right)\Bigg|$$

$$< 4|\max\{\mu^2, 1\} \max\{\mu'^2, 1\}|$$

$$+ \left|\max\left\{1, \frac{1}{\mu^2}\right\} - \max\left\{1, \frac{1}{\mu'^2}\right\}\right|$$

Thus $|\psi_\mu(s) - \psi_{\mu'}(s)|$ converges to zero as $\mu' \to \mu$ uniformly, i.e. the convergence is independent of $s$. Hence the assertion is obtained. □

Using the lemmas above we present the necessary and sufficient condition for the conic section to be optimal approximation of the planar curve with respect to the maximum norm (1).

**Theorem 2.** *The conic section $\mathbf{b}_{\mu^*}(t)$ having the weight $\mu^*$ is the optimal approximation of $\mathbf{p}(s)$ with respect to the maximum norm (1) if and only if the error function $\psi_{\mu^*}(s)$ is equiosculating two times in $[a,b]$, i.e., $\psi_{\mu^*}(s)$ has the points $s_1$ and $s_2$ in $(a,b)$ such that*

$$\|\psi_{\mu^*}(s)\|_{L^\infty} = \psi_{\mu^*}(s_1) = -\psi_{\mu^*}(s_2)$$

*Furthermore, $\mu^*$ exists and is unique.*

**Proof.** We adapt the idea from the earlier work by Eisele [5] for the proof of this theorem. Let $\psi_{\mu^*}(s)$ be equiosculating two times in $[a,b]$. Assume that there exists $\mu'$ such that

$$\|\psi_{\mu'}(s)\|_{L^\infty} \leq \|\psi_{\mu^*}(s)\|_{L^\infty}$$

Since $\psi_{\mu^*}(s)$ is equiosculating two times, and any error function $\psi_\mu(s)$ is continuous, there exists $s_0 \in (a,b)$ such that $\psi_{\mu^*}(s_0) = \psi_{\mu'}(s_0)$. By Lemma 1, $\mu' = \mu^*$. That is to

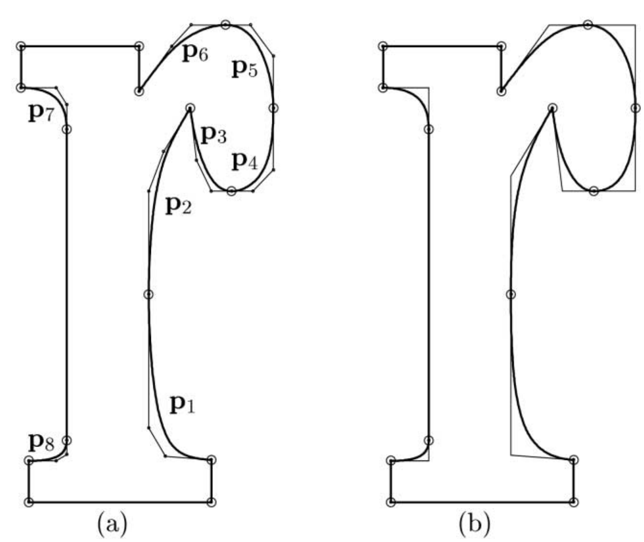


Fig. 3. (a) The outline of the font '**r**' constructed by cubic rational Bézier segments plotted as thick lines. The polygon shown by thin lines means the control-points of cubic (rational) Bézier curves and the circles are the junction points of two consecutive segments. (b) The optimal conic approximation of the font with respect to the maximum norm plotted as thick lines. The polygon means the control-points of the conic sections in form of quadratic rational curves, and the circles are the junction points of the conic sections.

Table 1
The control-points and weights of the cubic rational Bézier curves $\mathbf{p}_i(s)$, $i = 1, \ldots, 8$, and $\mathbf{p}_1^j(s)$, $j = 1, 2$ in standard form

| Segments | Control-points | Weights |
|---|---|---|
| $\mathbf{p}_1(s)$ | (212,48) (160,52) (141,84) (141,235) | (1,1.392,0.859,1) |
| $\mathbf{p}_2(s)$ | (141,235) (141,352) (158,397) (188,446) | (1,1.109,1.556,1) |
| $\mathbf{p}_3(s)$ | (188,446) (195,387) (212,352) (235,352) | (1,0.865,1.021,1) |
| $\mathbf{p}_4(s)$ | (235,352) (259,352) (282,376) (282,446) | (1,0.978,1.261,1) |
| $\mathbf{p}_5(s)$ | (282,446) (282,505) (256,540) (228,540) | (1,0.853,1.137,1) |
| $\mathbf{p}_6(s)$ | (228,540) (189,540) (167,516) (130,465) | (1,0.814,1.244,1) |
| $\mathbf{p}_7(s)$ | (−4,469) (36,469) (48,450) (48,422) | (1,1,1,1) |
| $\mathbf{p}_8(s)$ | (48,70) (48,54) (36,47) (5,47) | (1,1,1,1) |
| $\mathbf{p}_1^1(s)$ | (212,48) (181.65,50.34) (167.58,57.11) (158.06,82.17) | (1,1.162,1.093,1) |
| $\mathbf{p}_1^2(s)$ | (158.06,82.17) (147.39,110.27) (141,165.51) (141,235) | (1,0.967,0.903,1) |

say, $\mathbf{b}_{\mu^*}(t)$ is the optimal conic approximation of $\mathbf{p}(s)$ with respect to the maximum norm.

Conversely, assume that $\psi_\mu(s)$ is not equiosculating two times. Without loss of generality we assume that

$$\left|\max_{s\in[a,b]} \psi_\mu(s)\right| > \left|\min_{s\in[a,b]} \psi_\mu(s)\right|$$

i.e. $\varepsilon := \max \psi_\mu(s) + \min \psi_\mu(s)$ is positive. By Lemma 2, there exists $\delta > 0$ such that if $|\mu - \mu'| < \delta$, then

$$\|\psi_\mu(s) - \psi_{\mu'}(s)\|_{L^\infty} < \varepsilon$$

Putting $\mu' = \mu - \delta/2$, we have

$$\psi_\mu(s) - \varepsilon < \psi_{\mu'}(s) < \psi_\mu(s)$$

for all $s \in [a,b]$. Since $\psi_\mu(s) - \varepsilon \geq -\|\psi_\mu\|_{L^\infty}$ for all $s \in [a,b]$, we have

$$-\|\psi_\mu\|_{L^\infty} < \psi_{\mu'}(s) < \|\psi_\mu\|_{L^\infty}$$

for all $s \in [a,b]$, or $\|\psi_{\mu'}(s)\|_{L^\infty} < \|\psi_\mu\|_{L^\infty}$. That is to say, $\mathbf{b}_\mu(t)$ is not the optimal approximation with respect to the maximum norm.

The existence of the optimal conic approximation follows from the compactness argument, and the uniqueness follows from Lemma 1. □

## 3. Numerical example

In this section we apply our characterization theorem to approximate cubic rational spline curves by conic spline curves. As a numerical example, the outline of the font '**r**' consisted of cubic rational spline curves is approximated by conic spline curves as shown in Fig. 3. The work for approximation of the composite cubic Bézier curves by the composite quadratic Bézier curves was done by Cox and Harris [2]. The outline of the font '**r**' in Fig. 3a consists of seven straight line segments, six cubic rational curves $\mathbf{p}_i(s)$, $i = 1, \ldots, 6$, and two cubic curves $\mathbf{p}_i(s)$, $i = 7, 8$, of which control points and weights are listed in Table 1. Using our characterization theorem we find the optimal conic approximation $\mathbf{b}_\mu(t)$ for each cubic rational curve $\mathbf{p}_i(s)$ with respect to the maximum norm (1), and merge all the approximate conic sections with the remaining straight line segments, as shown in Fig. 3b.

Table 2
The optimal conic approximation $\mathbf{b}_\mu(t)$ for each cubic rational Bézier curve $\mathbf{p}_i(s)$, $i = 1, \ldots, 8$, and $\mathbf{p}_1^j(s)$, $j = 1, 2$ with respect to the maximum norm (1), and the uniform norm of the error function $\psi_\mu(s)$

| Segments | $\mathbf{b}_\mu(t)$ | $\|\psi_\mu(s)\|_{L^\infty}$ | $\dfrac{\|\psi_\mu(s)\|_{L^\infty}}{\lvert\mathbf{b}_0 + \mathbf{b}_2 - 2\mathbf{b}_1\rvert}$ |
|---|---|---|---|
| $\mathbf{p}_1(s)$ | $\mu = 1.4581$ | 3.894 | $2.051 \times 10^{-2}$ |
| $\mathbf{p}_2(s)$ | $\mu = 1.4595$ | 1.320 | $1.778 \times 10^{-2}$ |
| $\mathbf{p}_3(s)$ | $\mu = 0.8834$ | $4.837 \times 10^{-1}$ | $4.977 \times 10^{-3}$ |
| $\mathbf{p}_4(s)$ | $\mu = 0.9887$ | $8.079 \times 10^{-1}$ | $7.687 \times 10^{-3}$ |
| $\mathbf{p}_5(s)$ | $\mu = 0.7380$ | $9.197 \times 10^{-1}$ | $8.484 \times 10^{-3}$ |
| $\mathbf{p}_6(s)$ | $\mu = 1.3861$ | 1.461 | $1.928 \times 10^{-2}$ |
| $\mathbf{p}_7(s)$ | $\mu = 1.0579$ | $3.116 \times 10^{-2}$ | $4.446 \times 10^{-4}$ |
| $\mathbf{p}_8(s)$ | $\mu = 1.1339$ | $7.764 \times 10^{-3}$ | $1.592 \times 10^{-4}$ |
| $\mathbf{p}_1^1(s)$ | $\mu = 1.4412$ | $1.74 \times 10^{-1}$ | $4.23 \times 10^{-3}$ |
| $\mathbf{p}_1^2(s)$ | $\mu = 0.8787$ | $1.28 \times 10^{-1}$ | $1.96 \times 10^{-3}$ |

For each conic approximation, we find the uniform norm $\|\psi(s)\|_{L^\infty}$ of the error function, as shown in Table 2. Assuming the tolerance to be equal to 2 in this example, the first segment $\mathbf{p}_1(s)$ should be subdivided. Since the segment has not any cusp or inflection points, the segment is subdivided at the largest-distance point from the line joining two end points. We denote two subdivided cubic rational Bézier curves of $\mathbf{p}_1(s)$ by $\mathbf{p}_1^1(s)$ and $\mathbf{p}_1^2(s)$, whose control points and weights in standard rational Bézier form are listed in Table 1. The curves $\mathbf{p}_1^1(s)$ and $\mathbf{p}_1^2(s)$ are also approximated by the optimal conic approximation with respect to the maximum norm and the numerical error bounds are presented in Table 2. We also calculate the ratio of the uniform norm to the length $|\mathbf{b}_0 + \mathbf{b}_2 - 2\mathbf{b}_1|$ as shown in Table 2. Since the upper bound (1) overestimates the Hausdorff distance in case the weights widely differ from one, we can see that the relative errors $\|\psi(s)\|_{L^\infty}/|\mathbf{b}_0 + \mathbf{b}_2 - 2\mathbf{b}_1|$ in the cases except for $\mathbf{p}_1^1(s)$ are larger than others. All of the given cubic rational Bézier curves and the approximate conic sections obtained by our

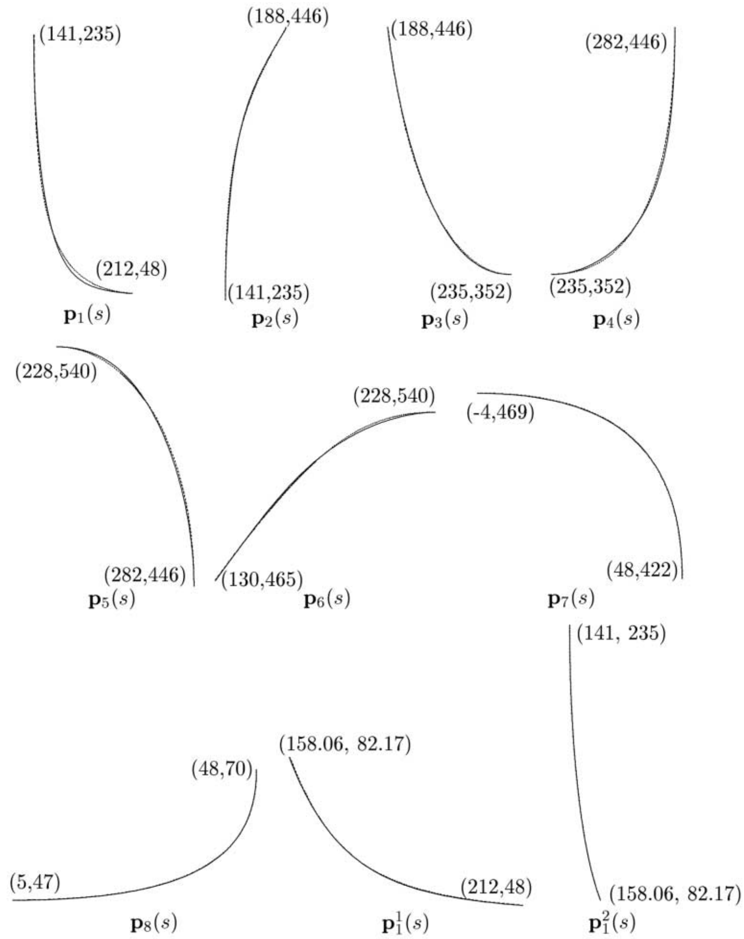


Fig. 4. The cubic rational Bézier curves $\mathbf{p}_i(s)$, $i = 1, \ldots, 8$, and $\mathbf{p}_1^j(s)$, $j = 1, 2$ plotted by solid lines with the same scale of the distance between two end points of each curve segment, and the optimal approximate conic sections $\mathbf{b}_\mu(t)$ by dashed lines, where $\mu$ is listed in Table 2.

characterization are plotted in Fig. 4. We also see the biased error of the approximation for $\mathbf{p}_1(s)$ and $\mathbf{p}_6(s)$. This is due to the difference of the lengths of two legs of the conic section, which is disregarded in formula (1) for the upper bound of the Hausdorff distance.

The method of minimization of the Hausdorff distance between the given planar curve and the approximate spline curve directly was proposed by Degen [3]. As shown in Table 3, we compare the numerical results of the conic approximation by our method to those of the method looking for the best approximation. Even if our method yields the conic approximations of which the Hausdorff distances are larger than those of the best approximation, it is not easy to find the closed form of the error function of the method looking for the best conic approximation, so that its algorithm is more complicated than that of our method.

Table 3
Two methods of conic approximations. The second and third column are numerical results for the weights and the Hausdorff distance by our method, respectively, and the fourth and fifth columns are those by the method looking for the best approximation

| Segments | $\mathbf{b}_\mu(t)$ | $d_H(\mathbf{b}_\mu, \mathbf{p})$ | Best approximation | Hausdorff distance |
|---|---|---|---|---|
| $\mathbf{p}_1(s)$ | $\mu = 1.4581$ | 2.36 | $\mu = 1.5422$ | 1.70 |
| $\mathbf{p}_2(s)$ | $\mu = 1.4595$ | $6.79 \times 10^{-1}$ | $\mu = 1.5225$ | $4.61 \times 10^{-1}$ |
| $\mathbf{p}_3(s)$ | $\mu = 0.8834$ | $2.21 \times 10^{-1}$ | $\mu = 0.8973$ | $2.94 \times 10^{-1}$ |
| $\mathbf{p}_4(s)$ | $\mu = 0.9887$ | $7.94 \times 10^{-1}$ | $\mu = 0.9687$ | $6.50 \times 10^{-1}$ |
| $\mathbf{p}_5(s)$ | $\mu = 0.7380$ | $7.17 \times 10^{-1}$ | $\mu = 0.7482$ | $6.11 \times 10^{-1}$ |
| $\mathbf{p}_6(s)$ | $\mu = 1.3861$ | $8.47 \times 10^{-1}$ | $\mu = 1.3375$ | $6.71 \times 10^{-1}$ |
| $\mathbf{p}_7(s)$ | $\mu = 1.0579$ | $2.71 \times 10^{-2}$ | $\mu = 1.0580$ | $2.66 \times 10^{-2}$ |
| $\mathbf{p}_8(s)$ | $\mu = 1.1339$ | $6.59 \times 10^{-2}$ | $\mu = 1.1342$ | $5.71 \times 10^{-3}$ |
| $\mathbf{p}_1^1(s)$ | $\mu = 1.4412$ | $1.12 \times 10^{-1}$ | $\mu = 1.4370$ | $1.02 \times 10^{-1}$ |
| $\mathbf{p}_1^2(s)$ | $\mu = 0.8787$ | $5.09 \times 10^{-2}$ | $\mu = 0.8819$ | $4.34 \times 10^{-2}$ |

## 4. Conclusions

In this paper we characterized the necessary and sufficient condition for the conic section to be optimal approximation of the given planar curve with respect to the maximum norm presented by Floater [8]. Using the characterization we presented the optimal conic approximation of the cubic rational Bézier curve, and gave the upper bound of the Hausdorff distance between two curves numerically. Although the error bound obtained by our method is larger than the error obtained by the method of minimization of the Hausdorff distance directly, the close form of the error function can be obtained in our method so that the algorithm is more simple than the other method.

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