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A rational quartic Bézier representation for conics

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Abstract

This paper presents a special representation for conic sections in the form of a rational quartic Bézier curve which has the same weight for all control points but the middle one. This representation allows a conic section to be joined with other conics in the same form or other integral B-spline curves in a way that the joined curve still possesses C^1 continuity in the homogeneous space, which is not possible if rational quadratic representation is adopted. This also allows the creation of skinned surfaces from section curves containing conic sections to possess better parametrization and curvature property. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Conic sections; Rational Bézier curve; Homogeneous space; Skinned surfaces

1. Introduction

Conic sections are widely used for computer-aided design and manufacturing in various industries. A significant reason for using NURBS for shape representation is their ability to represent conic sections precisely. Traditionally, conic sections, when represented by NURBS, are in the form of a rational quadratic Bézier curve. The standard form of this representation can be written as

$$\boldsymbol{r}(t) = \frac{B_{2,0}(t)\boldsymbol{p}_0 + B_{2,1}(t)w\boldsymbol{p}_1 + B_{2,2}(t)\boldsymbol{p}_2}{B_{2,0}(t) + B_{2,1}(t)w + B_{2,2}(t)}, \quad t \in [0,1], \tag{1}$$

where $\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2 \in \mathbb{R}^2$ are the control points, $w \in \mathbb{R}$ is the weight associated with $\boldsymbol{p}_1$, $B_{2,0}(t) = (1-t)^2$, $B_{2,1}(t) = 2t(1-t)$, $B_{2,2}(t) = t^2$ are the Bernstein basis functions. With this representation, the type of conic is characterized by the value of the middle weight, w: $\boldsymbol{r}(t)$ is an ellipse when $w < 1$, a parabola when $w = 1$ and a hyperbola when $w > 1$. Also, when w is negative, $\boldsymbol{r}(t)$ is the complementary segment of the original conic segment (see (Lee, 1987) and (Farin, 1993)).

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In the literature, conic sections have been extensively studied. Several rational representations for circular arcs, a special kind of conic section, are described in (Piegl and Tiller, 1989). A very detailed study on conic sections, in the form of a rational quadratic curve, is presented in (Lee, 1987). In (Wang and Wang, 1992), the necessary and sufficient conditions for representing conic sections by the rational cubic Bézier curve with proper parametrization are discussed. In (Chou, 1995), the possibility of using a single Bézier curve to represent a full circle is studied and it was proven that it requires at least a degree 5 Bézier curve in order to represent a full circle without resorting to negative weights. In (Sánchez-Reyes, 1997) it was proven that all Bézier circular arcs other than quadratic arcs are degenerate. In (Blanc and Schlick, 1996) a zigzag reparametrization (actually a quadratic rational reparametrization) is introduced to make a full circle, represented in multi-segmented quadratic NURBS, to be C^2 continuous or to be quasi-uniformly parametrized.

1.1. A common design problem

The rational quadratic Bézier form is an elegant way to represent conic sections; however, representing conic sections in rational quadratic Bézier form does not guarantee that two rational quadratic conics, though C^1 continuous at a common joint in the Euclidian space, will remain C^1 continuous in the homogeneous space. In fact, most of the time, they are only C^0 in the homogeneous space because of the different weights associated with the control points closest to the joint and at the joint (see Appendix A for a study on this continuity issue). This characteristic has caused the rise of a serious problem in the downstream surface creation using techniques such as skinning or lofting.

Fig. 1(a) shows four quadratic curves, each of which is formed by concatenating two conic sections. They are all C^1 continuous in the Euclidian space with a double knot at $t = 1/2$ (see Appendix B for the data for control points and weights). Using these four curves as section curves for skinned surface creation resulting in the surface shown in Fig. 1(b). The surface is not only badly parametrized, its isoparametric curves in the u-direction (the direction of the section curve) are C^0 continuous only. Therefore, despite the fact that all section curves are C^1 continuous, the resulting skinned surface exhibits creases and is generally C^0 continuous only! This is completely unacceptable from a designer's point of view.

This phenomenon, referred as surface crease problem thereafter, is a general issue with the current skinning or lofting surface creation. It could emerge whenever one or some of the section curves are rational and are not C^1 continuous in both the Euclidian and the homogenous spaces. Due to the rationality of section curves, the skinned surface construction is forced to take place in the homogeneous space. However, these section curves are C^0 only in the homogeneous space. Consequently, the constructed surface is also C^0 continuous in the u-direction and so are the u-direction isoparametric curves.

This problem was previously addressed in (Hohmeyer and Barsky, 1991) and (Tokuyama and Konno, 2001). Hohmeyer and Barsky used a *smoothing function* to improve the continuity of the given rational B-spline curves in the homogenous space. However, there is no general algorithm to determine the smoothing function and the resulting skinned surface might be raised to a higher degree. Tokuyama and Konno tried to reparametrize the given rational B-spline curves by solving a nonlinear equation set to achieve the C^1 continuity in the homogenous space. However, there is also no guarantee that the reparametrization will success. In fact, the reparametrization fails when the rational B-spline curve contains two 90-degree arcs. Furthermore, the resulting skinned surface might possess too many small patches since the knot vectors of the given section curves are altered by the reparametrization process.

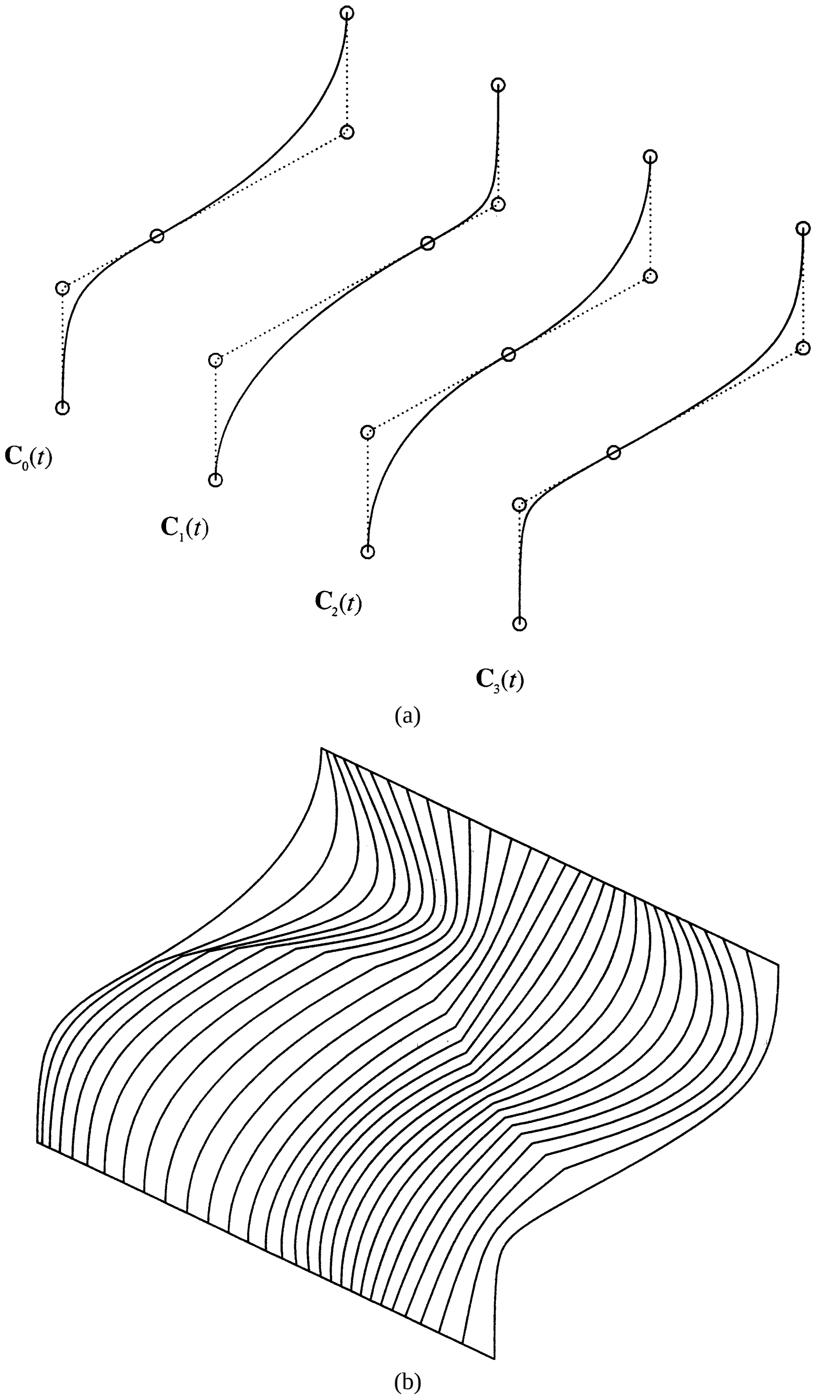


Fig. 1. The rational quadratic section curves and the resulting skinned surface. The surface is badly parametrized and the isoparametric curves exhibit C^1 discontinuity. (a) The four C^1 continuous section curves in the form of quadratic NURBS. (b) The resulting skinned surface.

1.2. The motivation

Instead of trying to solve the surface crease problem arising from skinning through general rational section curves, we focus our effort on the cases where the section curves contain only conic sections and integral Bézier curves. For such section curves, their C^1 discontinuities in the homogeneous space come from representing conics in the rational quadratic Bézier form. Therefore, a resolution is to look for a different rational Bézier representation for conic sections that do not make the section curve become C^0 in the homogeneous space. An easy way to achieve this goal is to force this new Bézier representation to have the same weight for the first two and the last two control points, which makes it at least degree 4. Therefore, our task becomes finding a representation for conic sections in the form of rational quartic Bézier curve, which has the same weights for all the control points except the middle one. If a conic section can be represented in this manner, joining a conic section with other conic sections in the same form or with other integral B-spline curves will result in a curve that is still C^1 continuous in the homogeneous space. This work is inspired by the work described in (Allen, 1993) where the study only focused on circular arcs, instead of general conic sections.

Although the proposed approach is only applicable when the section curves contain only conic sections and integral Bézier curves, given the fact that polynomial curves remain the dominant form for curves modeling and rational curves are mainly used to model conic sections (and circular arcs), this approach should suffice for practical purpose.

The rest of this paper is organized as follows. In Section 2 we obtain the representation for conic sections in the rational quartic Bézier form given its rational quadratic Bézier representation. The solutions are analyzed in Section 3. Some examples are given in Section 4 and conclusion is made in Section 5.

2. Rational quartic Bézier conic

2.1. Conversion from rational quadratic conic to rational quartic conic

Given a standard rational quadratic conic section described by (1), the goal of this paper is to see if we can represent this conic section as a rational quartic Bézier curve, written as

$$\bar{\boldsymbol{r}}(t) = \frac{B_{4,0}(t)\bar{\boldsymbol{p}}_0 + B_{4,1}(t)\bar{\boldsymbol{p}}_1 + B_{4,2}(t)w_2\bar{\boldsymbol{p}}_2 + B_{4,3}(t)\bar{\boldsymbol{p}}_3 + B_{4,4}(t)\bar{\boldsymbol{p}}_4}{B_{4,0}(t) + B_{4,1}(t) + B_{4,2}(t)w_2 + B_{4,3}(t) + B_{4,4}(t)} = \frac{\boldsymbol{R}(t)}{W(t)}, \quad t \in [0,1] \tag{2}$$

where $\bar{\boldsymbol{p}}_i$, $i = 0, \ldots, 4$, are the control points, w_2 is the weight associated with $\bar{\boldsymbol{p}}_2$, and $B_{4,i}(t) = \binom{4}{i}(1-t)^{4-i}t^i$, $i = 0, \ldots, 4$, are the Bernstein basis functions. Here, we reparametrize the weights so that all of them but the middle weight are 1.

For $\bar{\boldsymbol{r}}(t)$ to be geometrically equivalent to $\boldsymbol{r}(t)$ in (1), it is obvious that $\bar{\boldsymbol{r}}(t)$ needs to be G^1 continuous with $\boldsymbol{r}(t)$ at the end points. Therefore, we can represent the control points of $\bar{\boldsymbol{r}}(t)$ via the control points of $\boldsymbol{r}(t)$ as

$$\bar{\boldsymbol{p}}_0 = \boldsymbol{p}_0, \tag{3a}$$

$$\bar{\boldsymbol{p}}_1 = \boldsymbol{p}_0 + \alpha_0(\boldsymbol{p}_1 - \boldsymbol{p}_0), \tag{3b}$$

$$\bar{\boldsymbol{p}}_2 = L_0\boldsymbol{p}_0 + L_1\boldsymbol{p}_1 + L_2\boldsymbol{p}_2, \tag{3c}$$

$$\bar{\boldsymbol{p}}_3 = \boldsymbol{p}_2 - \alpha_1(\boldsymbol{p}_2 - \boldsymbol{p}_1), \quad \text{and} \tag{3d}$$
$$\bar{\boldsymbol{p}}_4 = \boldsymbol{p}_2, \tag{3e}$$

where α_0 and α_1 are arbitrary constants, $L_i,\ i = 0, 1, 2$, are the barycentric coordinates of $\bar{\boldsymbol{p}}_2$ with respect to the triangle $\Delta \boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{p}_2$ and $L_0 + L_1 + L_2 = 1$.

Substituting Eqs. (3a)–(3e) into (2), we have

$$\bar{\boldsymbol{r}}(t) = \bar{\tau}_0(t)\boldsymbol{p}_0 + \bar{\tau}_1(t)\boldsymbol{p}_1 + \bar{\tau}_2(t)\boldsymbol{p}_2 \tag{4}$$

where

$$\bar{\tau}_0(t) = \frac{1}{W(t)}\big((1-t)^4 + 4t(1-t)^3(1-\alpha_0) + 6w_2t^2(1-t)^2L_0\big),$$
$$\bar{\tau}_1(t) = \frac{1}{W(t)}\big(4t(1-t)^3\alpha_0 + 6w_2t^2(1-t)^2L_1 + 4t^3(1-t)\alpha_1\big), \quad \text{and}$$
$$\bar{\tau}_2(t) = \frac{1}{W(t)}\big(6w_2t^2(1-t)^2L_2 + 4t^3(1-t)(1-\alpha_1) + t^4\big).$$

An interesting property of $\boldsymbol{r}(t)$ is that it remains unchanged when we swap the location of $\boldsymbol{p}_0$ and $\boldsymbol{p}_2$ and replace t by $1-t$. We would like $\bar{\boldsymbol{r}}(t)$ to possess the same property, namely,

$$\bar{\boldsymbol{r}}(t) = \bar{\tau}_0(1-t)\boldsymbol{p}_2 + \bar{\tau}_1(1-t)\boldsymbol{p}_1 + \bar{\tau}_2(1-t)\boldsymbol{p}_0. \tag{5}$$

From (4) and (5), we can deduce that $\alpha_0 = \alpha_1$ and $L_0 = L_2$. Note that Eq. (5) is not a necessary condition for $\bar{\boldsymbol{r}}(t)$ to be a conic section; however, applying it will enforce our solutions to possess nice "symmetry" properties, e.g., $\bar{\boldsymbol{p}}_2$ will lie on the line connecting $\boldsymbol{p}_1$ and $\frac{1}{2}(\boldsymbol{p}_0 + \boldsymbol{p}_2)$ and line $\bar{\boldsymbol{p}}_1\bar{\boldsymbol{p}}_3$ will always be parallel to line $\boldsymbol{p}_0\boldsymbol{p}_2$.

Letting $\alpha_0 = \alpha_1 = \alpha$ and $L_0 = L_2 = (1 - L_1)/2$, the barycentric coordinates of $\bar{\boldsymbol{r}}(t)$ become

$$\bar{\tau}_0(t) = \frac{1}{W(t)}\big((1-t)^4 + 4t(1-t)^3(1-\alpha) + 3w_2t^2(1-t)^2(1-L_1)\big), \tag{6a}$$
$$\bar{\tau}_1(t) = \frac{1}{W(t)}\big(4t(1-t)^3\alpha + 6w_2t^2(1-t)^2L_1 + 4t^3(1-t)\alpha\big), \quad \text{and} \tag{6b}$$
$$\bar{\tau}_2(t) = \frac{1}{W(t)}\big(3w_2t^2(1-t)^2(1-L_1) + 4t^3(1-t)(1-\alpha) + t^4\big). \tag{6c}$$

Furthermore, it is well known that for any point on a conic section $\boldsymbol{r}(t)$, its barycentric coordinates $\tau_0,\ \tau_1,\ \tau_2$, where $\tau_0 + \tau_1 + \tau_2 = 1$, with respect to the triangle $\Delta \boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{p}_2$, satisfies

$$f\big(\boldsymbol{r}(t)\big) = \tau_1^2(t) - 4w^2\tau_0(t)\tau_2(t) = 0. \tag{7}$$

Therefore, the barycentric coordinates of any point on $\bar{\boldsymbol{r}}(t)$ with respect to the triangle $\Delta \boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{p}_2$ should also satisfy

$$f\big(\bar{\boldsymbol{r}}(t)\big) = \bar{\tau}_1^2(t) - 4w^2\bar{\tau}_0(t)\bar{\tau}_2(t) = 0. \tag{8}$$

After substituting (6a), (6b) and (6c) into (8) and performing some tedious algebra manipulation using Maple, a software for symbolic mathematic manipulation, we obtain a degree 8 rational polynomial of t

$$f\big(\bar{\boldsymbol{r}}(t)\big) = \frac{1}{W^2(t)}\sum_{i=0}^{8} c_i t^i, \tag{9}$$

where

$$\begin{aligned}
c_0 &= 0,\\
c_1 &= 0,\\
c_2 &= 16\alpha^2 + 12w^2(L_1-1)w_2,\\
c_3 &= -96\alpha^2 + 16w^2(\alpha-1) - 24\big(w^2(2\alpha+1)(L_1-1) - 2\alpha L_1\big)w_2,\\
c_4 &= 272\alpha^2 - 4w^2\big(16\alpha^2 - 12\alpha - 3\big) + 60\big(w^2(4\alpha-1)(L_1-1) - 4\alpha L_1\big)w_2\\
&\quad - 36\big(w^2(L_1-1)^2 - L_1^2\big)w_2^2,\\
c_5 &= -448\alpha^2 + 16w^2\big(16\alpha^2 - 21\alpha + 6\big) - 48\big(w^2(11\alpha-6)(L_1-1) - 11\alpha L_1\big)w_2\\
&\quad + 144\big(w^2(L_1-1)^2 - L_1^2\big)w_2^2,\\
c_6 &= 448\alpha^2 - 8w^2(6\alpha-5)(8\alpha-5) + 48\big(w^2(13\alpha-9)(L_1-1) - 13\alpha L_1\big)w_2\\
&\quad - 216\big(w^2(L_1-1)^2 - L_1^2\big)w_2^2,\\
c_7 &= -256\alpha^2 + 16w^2(4\alpha-3)^2 - 96\big(w^2(4\alpha-3)(L_1-1) - 4\alpha L_1\big)w_2\\
&\quad + 144\big(w^2(L_1-1)^2 - L_1^2\big)w_2^2,\\
c_8 &= 64\alpha^2 - 4w^2(4\alpha-3)^2 + 24\big(w^2(4\alpha-3)(L_1-1) - 4\alpha L_1\big)w_2 - 36\big(w^2(L_1-1)^2 - L_1^2\big)w_2^2.
\end{aligned}$$

Requiring $f(\bar{\boldsymbol r}(t))$ to be zero means all of its coefficients $c_i,\ i = 0,\dots,8$, must be zero. From $c_2 = 0$, we obtain

$$w_2 = \frac{4\alpha^2}{3w^2(1-L_1)}. \tag{10}$$

Substituting (10) into $c_3 = 0$ results in

$$c_3 = 16w^2(\alpha-1) + 64\alpha^2(\alpha-1) - \frac{64\alpha^3 L_1}{w^2(L_1-1)} = 0, \tag{11}$$

thus

$$L_1 = \frac{w^2(\alpha-1)(w^2+4\alpha^2)}{w^2(\alpha-1)(w^2+4\alpha^2) - 4\alpha^3}. \tag{12}$$

Substituting w_2 and L_1 in (10) and (12) into $c_i,\ i = 4,\dots,8$, we obtain

$$\begin{aligned}
c_4 &= 64\left(1 - \frac{1}{w^2}\right)\alpha^4 - 128\alpha^3 - 32\big(w^2-3\big)\alpha^2 + 64w^2\alpha + \left(4w^4 - 36w^2 - \frac{8w^4}{\alpha} + \frac{4w^4}{\alpha^2}\right), \qquad (13)\\
c_5 &= -4c_4, \quad c_6 = 6c_4, \quad c_7 = -4c_4 \quad \text{and} \quad c_8 = c_4.
\end{aligned}$$

Therefore, if we can find an α satisfying (13), all the coefficients of $f(\bar{\boldsymbol r}(t))$ are zero and the solution for $\bar{\boldsymbol r}(t)$ is found. When $w \neq 1,\ c_4$ can be factored as

$$c_4 = \frac{4}{w^2\alpha^2}\big((w+1)\alpha - w\big)\big((w-1)\alpha - w\big)(2\alpha - w)^2(2\alpha + w)^2 = 0, \tag{14}$$

from which the four roots for α are

$$-\frac{w}{2}, \quad \frac{w}{2}, \quad \frac{w}{w-1} \quad \text{and} \quad \frac{w}{w+1}.$$

Table 1
Different w_2 and L_1 values for different α

α	w_2	L_1
$\frac{w}{2}$	$-\frac{1}{3}(2w^2-4w-1)$	$\frac{2w(w-2)}{2w^2-4w-1}$
$-\frac{w}{2}$	$-\frac{1}{3}(2w^2+4w-1)$	$\frac{2w(w+2)}{2w^2+4w-1}$
$\frac{w}{w+1}$	$\frac{1}{3}\frac{w^2+w+4}{w+1}$	$\frac{w(w^2+2w+5)}{(w+1)(w^2+w+4)}$
$\frac{w}{w-1}$	$-\frac{1}{3}\frac{w^2-w+4}{w-1}$	$\frac{w(w^2-2w+5)}{(w-1)(w^2-w+4)}$

When $w=1$ where $\boldsymbol{r}(t)$ is a parabola,

$$c_4=-128\alpha^3+64\alpha^2+64\alpha-32-\frac{8}{\alpha}+\frac{4}{\alpha^2}=-\frac{4}{\alpha^2}(2\alpha+1)^2(2\alpha-1)^3=0 \tag{15}$$

and the two solutions for α are $1/2$ and $-1/2$, which are part of the roots found when $w\neq 1$.

Substituting each solution for α into Eqs. (10) and (12), we can find the corresponding w_2 and L_1, thus the rational quartic Bézier conic section $\bar{\boldsymbol{r}}(t)$. Table 1 lists the different w_2 and L_1 values for different α. Note that above equations are derived based on standard rational quadratic conics. Conversion of non-standard rational quadratic conics can be done by first converting them to standard form and then applying above equations.

The rational quartic Bézier conic section resulting from $\alpha=w/(w+1)$ can also be obtained from the result presented in (Blanc and Schlick, 1996) where the rational quadratic Bézier conic is reparametrized by a rational quadratic polynomial, characterized by a scalar variable p. However, there are two aspects that distinct the presented article with the work by Blanc:

- This paper restricted the rational quartic Bézier curve to take the special form as depicted by (2) and solves the implicit equation of conic sections (Eq. (8)) directly. Therefore, we are able to find the other solutions which are not shown in (Blanc and Schlick, 1996).
- Our goal is to resolve (or at least alleviate) the surface crease problem when creating skinned surfaces from section curves containing conic sections. Blanc used their result to obtain a multi-segment quartic NURBS representation for circular arcs, which are C^2 continuous or quasi-uniformly parameterized. However, the C^2 continuity in the Euclidian space does not guarantee the smoothness of the skinned surfaces created from them since they are still C^0 only in the homogeneous space.

2.2. The semi-circle and semi-ellipse

Since semi-circles or semi-ellipses cannot be represented by (1) without using positive weights and finite control points, we will show that rational quartic Bézier form is capable of representing semi-circles or semi-ellipses without resorting to zero weights by following the same example given in (Farin, 1993).

Let a conic be given by $\boldsymbol{p}_0=[-1,0]^{\mathrm{T}}$, $\boldsymbol{p}_2=[1,0]^{\mathrm{T}}$, $\boldsymbol{p}_1=[0,\tan\theta]^{\mathrm{T}}$ and the middle weight $w=c\cos\theta$, where $\theta=\angle \boldsymbol{p}_1\boldsymbol{p}_0\boldsymbol{p}_2$ and c is an arbitrary constant. When converting this conic from quadratic form to quartic form, it can be written as

$$\boldsymbol{c}(t) = \frac{(1-t)^4\begin{bmatrix}-1\\0\end{bmatrix} + 4t(1-t)^3\begin{bmatrix}\alpha-1\\\alpha\tan\theta\end{bmatrix} + 6w_2t^2(1-t)^2\begin{bmatrix}0\\L_1\tan\theta\end{bmatrix}}{(1-t)^4+4t(1-t)^3+6w_2t^2(1-t)^2+4t^3(1-t)+t^4}$$

$$+\frac{4t^3(1-t)\begin{bmatrix}1-\alpha\\\alpha\tan\theta\end{bmatrix} + t^4\begin{bmatrix}1\\0\end{bmatrix}}{(1-t)^4+4t(1-t)^3+6w_2t^2(1-t)^2+4t^3(1-t)+t^4}. \tag{16}$$

Substituting $\alpha = w/(w+1)$ and corresponding L_1 and w_2 to $\boldsymbol{c}(t)$ and let θ approach $\pi/2$, $\boldsymbol{c}(t)$ becomes

$$\boldsymbol{c}(t) = \frac{(1-t)^4\begin{bmatrix}-1\\0\end{bmatrix} + 4t(1-t)^3\begin{bmatrix}-1\\c\end{bmatrix} + 8t^2(1-t)^2\begin{bmatrix}0\\\frac{5}{4}c\end{bmatrix} + 4t^3(1-t)\begin{bmatrix}1\\c\end{bmatrix} + t^4\begin{bmatrix}1\\0\end{bmatrix}}{(1-t)^4+4t(1-t)^3+8t^2(1-t)^2+4t^3(1-t)+t^4} \tag{17}$$

whose control points are

$$\begin{bmatrix}x\\y\\w\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\c\\1\end{bmatrix}, \begin{bmatrix}0\\\frac{5}{4}c\\\frac{4}{3}\end{bmatrix}, \begin{bmatrix}1\\c\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}.$$

Substituting $\alpha = w/2$ and corresponding L_1 and w_2 to $\boldsymbol{c}(t)$ and let θ approaches $\pi/2$, we obtain

$$\boldsymbol{c}(t) = \frac{(1-t)^4\begin{bmatrix}-1\\0\end{bmatrix} + 4t(1-t)^3\begin{bmatrix}-1\\\frac{1}{2}c\end{bmatrix} + 2t^2(1-t)^2\begin{bmatrix}0\\4c\end{bmatrix} + 4t^3(1-t)\begin{bmatrix}1\\\frac{1}{2}c\end{bmatrix} + t^4\begin{bmatrix}1\\0\end{bmatrix}}{(1-t)^4+4t(1-t)^3+2t^2(1-t)^2+4t^3(1-t)+t^4} \tag{18}$$

whose control points are

$$\begin{bmatrix}x\\y\\w\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\\frac{1}{2}c\\1\end{bmatrix}, \begin{bmatrix}0\\4c\\\frac{1}{3}\end{bmatrix}, \begin{bmatrix}1\\\frac{1}{2}c\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}.$$

For $c = 1$, we obtain a unit semicircle; other values of c result in semi-ellipses. We can see that in either Eq. (17) or (18), all the weights are positive, thus preserving the convex hull property.

3. Analysis

In the previous section we had shown that there are generally four solutions for $\bar{\boldsymbol{r}}(t)$ for a given value of w. Since Eq. (8) is the implicit form of the conic section, it is perceivable that some solutions could lead to the complementary segment of $\boldsymbol{r}(t)$ instead of $\boldsymbol{r}(t)$ itself. Therefore, we should limit the desired solution $\bar{\boldsymbol{r}}(t)$ to be on the same side of triangle $\Delta\boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{p}_2$ as $\boldsymbol{r}(t)$ is. Furthermore, the desired solution $\bar{\boldsymbol{r}}(t)$ should also contain nonnegative weights only since negative weights will destroy the convex hull property of NURBS curves.

Whether $\boldsymbol{r}(t)$ or $\bar{\boldsymbol{r}}(t)$ is inside or outside the triangle $\Delta\boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{p}_2$ can be easily judged from the direction of its first derivative at $t = 0$. For rational quadratic conics, we have $\boldsymbol{r}'(t=0) = 2w(\boldsymbol{p}_1 - \boldsymbol{p}_0)$. Therefore, $\boldsymbol{r}(t)$ lies inside triangle $\Delta\boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{p}_2$ when w is positive and outside the triangle when w is negative. Similarly, we have $\bar{\boldsymbol{r}}'(t=0) = 4(\bar{\boldsymbol{p}}_1 - \bar{\boldsymbol{p}}_0) = 4\alpha(\boldsymbol{p}_1 - \boldsymbol{p}_0)$ for the rational quartic Bézier conics. Therefore, $\bar{\boldsymbol{r}}(t)$ is inside triangle $\Delta\boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{p}_2$ when α is positive and outside the triangle when α is negative.

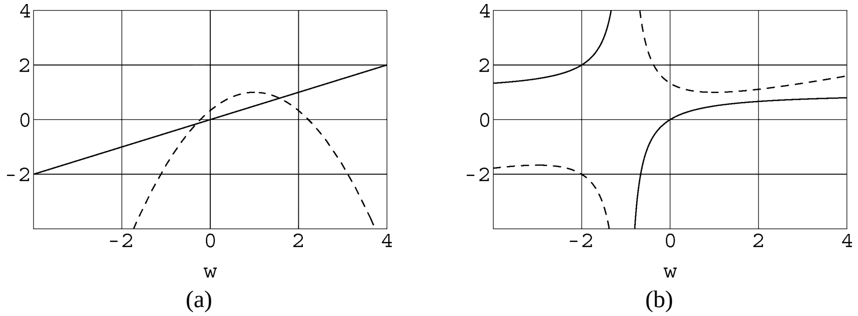


Fig. 2. The α (in solid line) and w_2 (in dashed line) as functions of w. (a) $\alpha = w/2$. (b) $\alpha = w/(w+1)$.

For the solution $\alpha = w/2$, the resulting $\bar{\boldsymbol{r}}(t)$ will always be on the same side of triangle $\Delta \boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{p}_2$ as $\boldsymbol{r}(t)$ is since α and w always have the same sign. However, w_2 is nonnegative only when $(2-\sqrt{6})/2 \leqslant w \leqslant (2+\sqrt{6})/2$ (as shown in Fig. 2(a)), which means this solution is desirable only when w falls in this range. For the solution $\alpha = w/(w+1)$, α and w have the same sign and w_2 is positive as long as $w > -1$ (as shown in Fig. 2(b)). This means that we can represent a much broader range of conic sections (with all positive weights) using $w/(w+1)$ than using $w/2$.

The solutions $w/(w-1)$ and $-w/2$ can be obtained by replacing w by $-w$ in $w/(w+1)$ and $w/2$. This means that the curves resulting from $w/(w-1)$ and $-w/2$ are simply the complementary segments of the curves resulting from $w/(w+1)$ and $w/2$ correspondingly.

For convenience, we denote, thereafter, the solution resulting from $\alpha = w/2$ and $\alpha = w/(w+1)$ as $\bar{\boldsymbol{r}}_1(t)$ and $\bar{\boldsymbol{r}}_2(t)$ respectively. A few notes are in order.

- $\bar{\boldsymbol{r}}_1(t)$ has exactly the same parametrization as the rational quadratic conic. This can be proven by substituting $\alpha = w/2$ and corresponding L_1 and w_2 into Eqs. (6a)–(6c). We find that the barycentric coordinates of $\bar{\boldsymbol{r}}_1(t)$ have a common factor $H(t)$ in the numerators and denominators as shown below:

$$\bar{\tau}_0(t) = \frac{(1-t)^2H(t)}{((1-t)^2+2wt(1-t)+t^2)H(t)} = \frac{(1-t)^2}{(1-t)^2+2wt(1-t)+t^2}, \tag{19a}$$

$$\bar{\tau}_1(t) = \frac{2wt(1-t)H(t)}{((1-t)^2+2wt(1-t)+t^2)H(t)} = \frac{2wt(1-t)}{(1-t)^2+2wt(1-t)+t^2}, \tag{19b}$$

$$\bar{\tau}_2(t) = \frac{t^2H(t)}{((1-t)^2+2wt(1-t)+t^2)H(t)} = \frac{t^2}{(1-t)^2+2wt(1-t)+t^2}, \tag{19c}$$

  where $H(t) = (2w-2)t^2 - (2w-2)t + 1$.
  After canceling out $H(t)$, the rational quartic Bézier form is reduced to the rational quadratic form.
- The emergence of $H(t)$ in both the numerators and denominators of (19) suggests that it is acting as a smoothing function that scales $\boldsymbol{r}^w(t)$, the homogeneous counterpart of $\boldsymbol{r}(t)$, up or down the four-dimensional cone defined by $\boldsymbol{r}(t)$ so that $H(t)\boldsymbol{r}^w(t)$ could acquire C^1 continuity in the homogeneous space. Since both $\boldsymbol{r}^w(t)$ and $H(t)\boldsymbol{r}^w(t)$ are on the four-dimensional cone, their projections to the $w = 1$ plane, i.e., $\boldsymbol{r}(t)$ and $\bar{\boldsymbol{r}}_1(t)$, are exactly the same.

- $\bar{\boldsymbol{r}}_2(t)$ can also be obtained by reparametrizing the original rational quadratic Bézier form by a rational quadratic polynomial $S(t)$, where

$$S(t) = \frac{pt + (1-p)t^2}{1 - 2(1-p)t + 2(1-p)t^2} \quad \text{with } p = \frac{2}{1+w}. \tag{20}$$

  This is a direct result from (Blanc and Schlick, 1996). The resulting barycentric coordinates are

$$\bar{\tau}_0(t) = \frac{(1-t)^2((w-1)t - (w+1))^2}{2(w+1)(w-1)^2t^2(1-t)^2 + (w+1)^2}, \tag{21a}$$

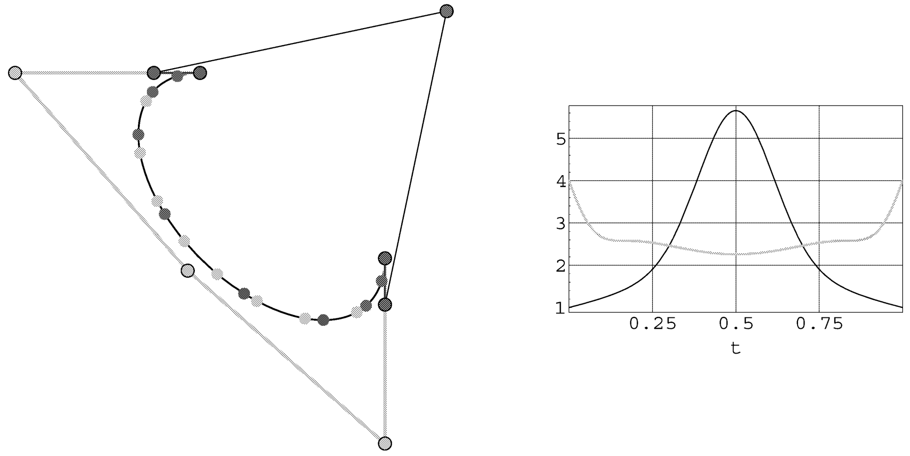
$$\bar{\tau}_1(t) = \frac{-2wt(1-t)((w-1)t+2)((w-1)t-(w+1))}{2(w+1)(w-1)^2t^2(1-t)^2 + (w+1)^2}, \tag{21b}$$

$$\bar{\tau}_2(t) = \frac{t^2((w-1)t+2)^2}{2(w+1)(w-1)^2t^2(1-t)^2 + (w+1)^2}. \tag{21c}$$

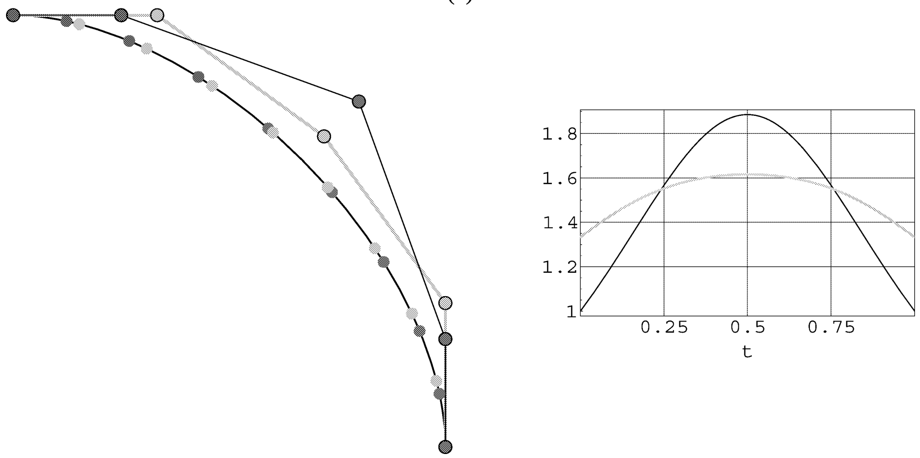
- When $-1 < w < 0, \bar{\boldsymbol{r}}_2(t)$ is the complementary segment of the elliptical arc inside triangle $\Delta \boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{p}_2$ and the middle weight for $\bar{\boldsymbol{r}}_2(t)$ is always positive. As $\angle \boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{p}_2$ approaches 180 degree, $\bar{\boldsymbol{r}}_2(t)$ approaches a full ellipse with all weights positive. However, $\bar{\boldsymbol{r}}_2(t)$ is never able to represent a full ellipse. This finding matches to the statements in (Chou, 1995) that it is impossible to have a quartic Bézier full circle with all positive weights.
- According to (Sederberg, 1986), $\bar{\boldsymbol{r}}_2(t)$ is "improperly parametrized". However, as bad as the term "improperly parametrized" may sound, the parametrization of $\bar{\boldsymbol{r}}_2(t)$ is generally better than that of $\bar{\boldsymbol{r}}_1(t)$ and the rational quadratic counterpart, as shown in next section.
- In (Sánchez-Reyes, 1997), it has been proven that "all Bézier circular arcs other than quadratic arcs are degenerate, that is, improperly parameterized and/or degree-reducible". Since the proof, utilizing the concept of algebraic geometry, is applicable to any planar rational parametric curves that admit an implicit polynomial equation $f(x, y) = 0$, the above statement should also be applicable for general conic sections. Although without providing a more rigorous proof, this conjecture is supported by the fact that both $\bar{\boldsymbol{r}}_1(t)$ and $\bar{\boldsymbol{r}}_2(t)$ are either improperly parameterized or degree-reducible to the rational quadratic form.

4. Examples

Figs. 3(a)–(f) show the different rational quartic Bézier conics with different value of w. All curves are produced based on the same base triangle: $\boldsymbol{p}_0 = [1, 0]$, $\boldsymbol{p}_1 = [1, 1]$ and $\boldsymbol{p}_2 = [0, 1]$. Fig. 3(g) shows the rational quartic Bézier semi-circle. Control polygons of $\bar{\boldsymbol{r}}_1(t)$ and $\bar{\boldsymbol{r}}_2(t)$ are displayed in dark gray and light gray respectively. Points are sampled at equal parametric values along the curve and are displayed in the same manner as for the control polygon, i.e., dark grayed points for $\bar{\boldsymbol{r}}_1(t)$ and light grayed points for $\bar{\boldsymbol{r}}_2(t)$. The distribution of the slope's magnitude is also shown in the right side of the conics. Note that the parametrization of $\bar{\boldsymbol{r}}_1(t)$ is exactly the same as $\boldsymbol{r}(t)$. It can also be seen that $\bar{\boldsymbol{r}}_2(t)$ generally has a more uniform parametrization than $\bar{\boldsymbol{r}}_1(t)$ and the rational quadratic conic.



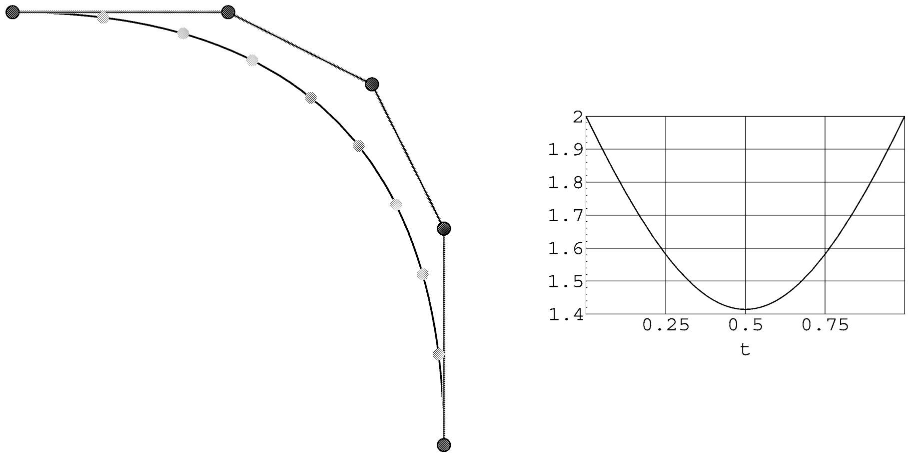
(a)



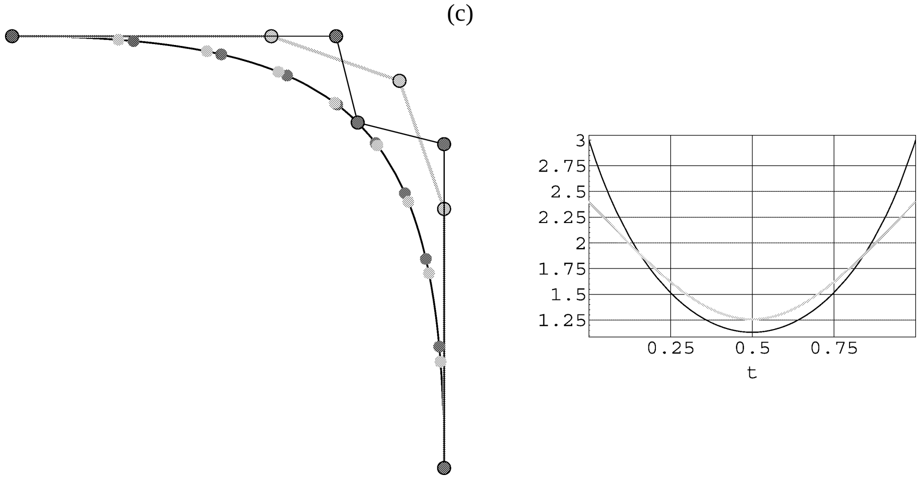
(b)

Fig. 3. Some quartic Bézier conics with different value of w. $\bar{\boldsymbol{r}}_1(t)$, the conic resulting from $\alpha = w/2$ is shown in dark gray. $\bar{\boldsymbol{r}}_2(t)$, the conic resulting from $\alpha = w/(w+1)$ is shown in light gray. (a) $w = -0.5$: The conic shown here is an elliptical arc with angular span 270 degree. Note that $\bar{\boldsymbol{r}}_1(t)$ no longer possesses the convex hull property due to the negative middle weight. (b) $w = 0.5$: The conic shown here is the complement of the elliptical arc shown in (a). (c) $w = 1.0$: $\bar{\boldsymbol{r}}_1(t)$ and $\bar{\boldsymbol{r}}_2(t)$ are the same curve, a parabola. (d) $w = 1.5$: The middle control point of $\bar{\boldsymbol{r}}_1(t)$ happens to coincide with its parametric midpoint. (e) $w = 2.0$: The control points of $\bar{\boldsymbol{r}}_1(t)$ are $[1,0]$, $[1,1]$, $[0.5,0.5]$, $[1,1]$, $[0,1]$, quite an interesting configuration. (f) $w = 2.5$: Again, $\bar{\boldsymbol{r}}_1(t)$ has negative middle weight because w is outside the useful range described in Section 3. (g) The rational quartic Bézier semi-circle.

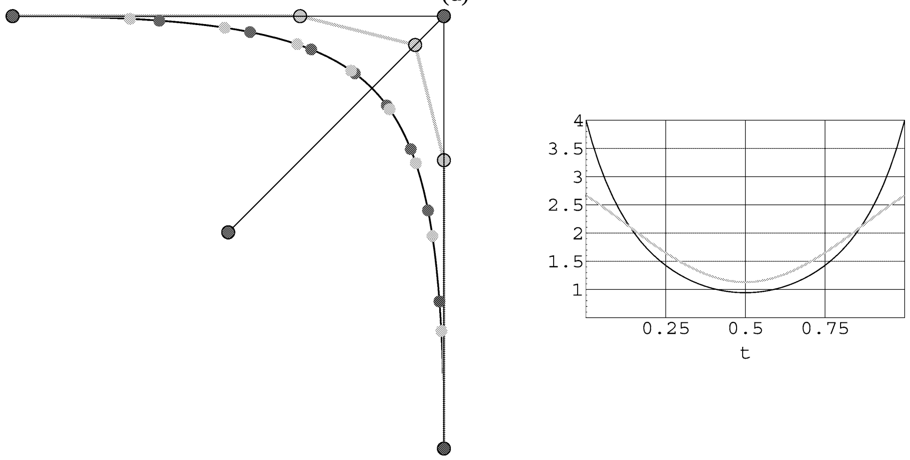
Fig. 4 shows the same section curves as in Fig. 1 and the resulting skinned surface. The difference is that conic sections are converted into rational quartic Bézier curves (using the $\alpha = w/(w+1)$ solution) for concatenation. The section curves are now C^1 continuous at the junction in the homogeneous space. Notice that the control points at the junctions no longer exist because of the knot removal performed in the curve concatenation process. As a result, the skinned surface possesses much better parametrization and continuity.



(c)

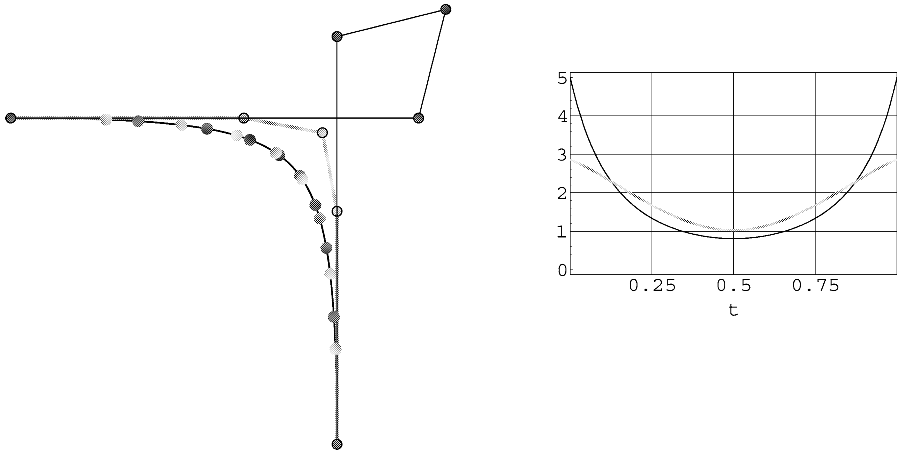


(d)

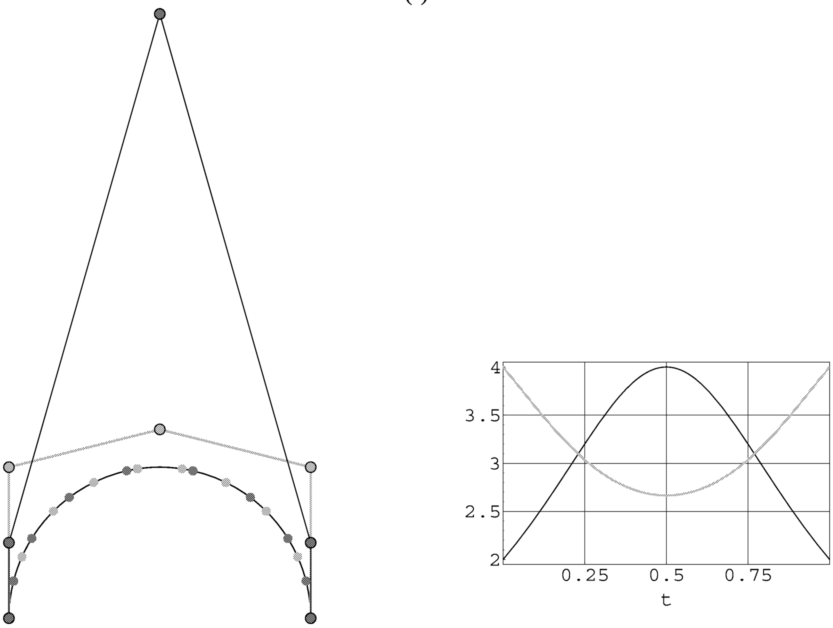


(e)

Fig. 3. (Continued).



(f)



(g)

Fig. 3. (Continued).

5. Conclusion

A rational quartic Bézier representation for conic sections is presented. This rational quartic Bézier curve has the same weight for all control points but the middle one, thus allowing the conic section to be joined with other conic sections (in the same representation) or integral B-spline curves with C^1 continuity in the homogeneous space. This will allow skinned surfaces created from section curves containing conic sections in the downstream operations to possess better parametrization and curvature property.

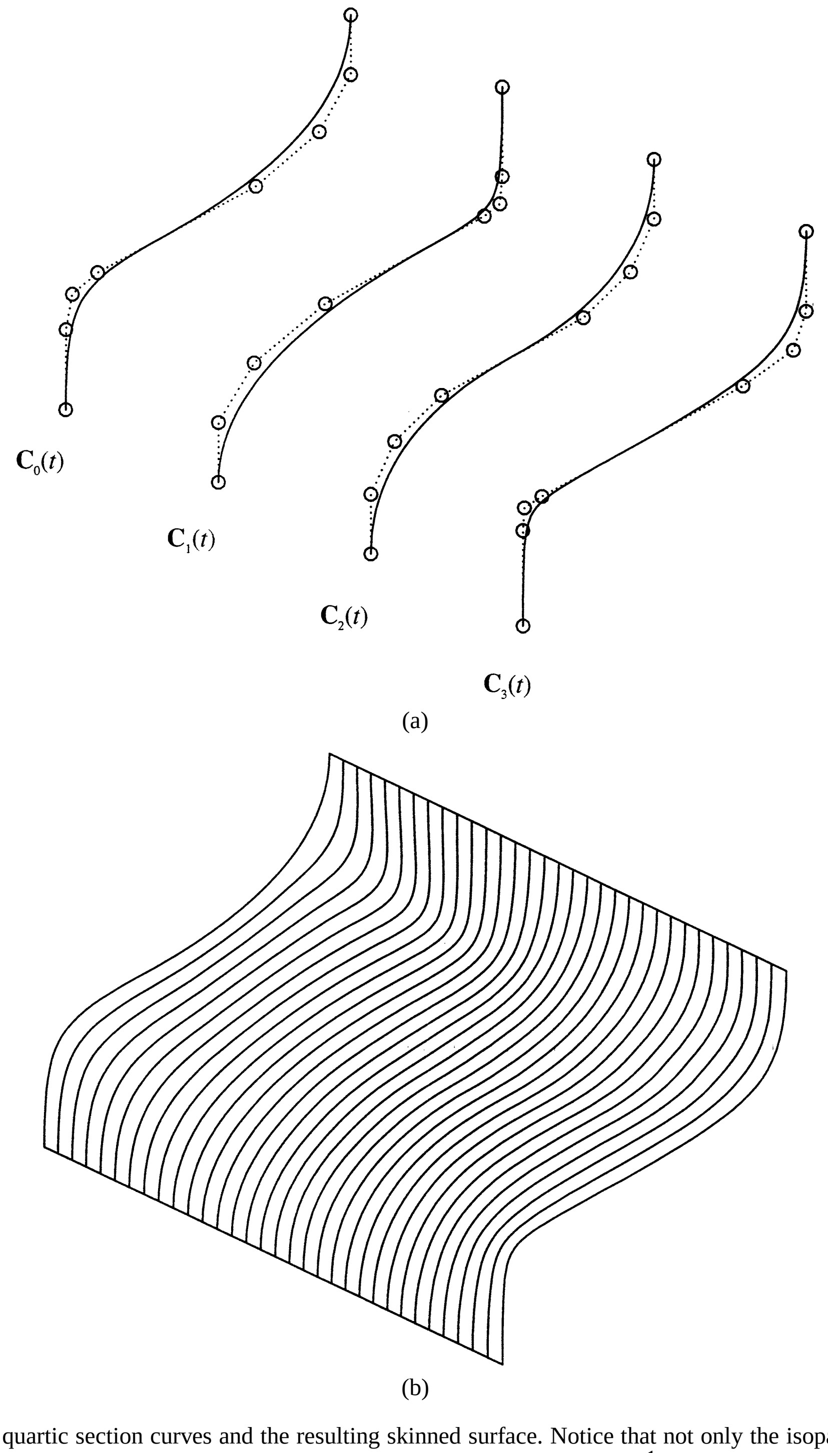


Fig. 4. The rational quartic section curves and the resulting skinned surface. Notice that not only the isoparametric curves are C^1 continuous, the surface's parametrization is greatly improved as well. (a) The four C^1 continuous section curves in the form of quartic NURBS. (b) The resulting skinned surface.

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Appendix A. Study of the continuity

Given two rational quadratic Bézier conics $\boldsymbol{c}_1(t)$ and $\boldsymbol{c}_2(t)$, where $\boldsymbol{c}_1(t)$ is defined by $\boldsymbol{P}_i$ and w_i, $i = 0, 1, 2$, $\boldsymbol{c}_2(t)$ by $\boldsymbol{P}_i$ and w_i, $i = 2, 3, 4$, and w_i are the weights associated with $\boldsymbol{P}_i$, the joined curve $\boldsymbol{C}(t)$ can be represented as a rational quadratic B-spline curve with the same set of control points, weights and knot vector $[0, 0, 0, h, h, 1, 1, 1]$. Note that the value of h changes only the parametrization but not the shape of $\boldsymbol{C}(t)$. The C^1 continuity of $\boldsymbol{C}(t)$ at $t = h$ requires

$$\frac{2w_1}{w_2 h}(\boldsymbol{P}_2 - \boldsymbol{P}_1) = \frac{2w_3}{w_2(1-h)}(\boldsymbol{P}_3 - \boldsymbol{P}_2), \tag{A.1}$$

which results in

$$\boldsymbol{P}_2 = \alpha \boldsymbol{P}_3 + (1-\alpha)\boldsymbol{P}_1, \quad 0 < \alpha < 1, \tag{A.2a}$$

$$h = \frac{\alpha w_1}{\alpha w_1 + (1-\alpha) w_3}. \tag{A.2b}$$

The C^1 continuity of $\boldsymbol{C}^w(t)$, the counterpart of $\boldsymbol{C}(t)$ in the homogeneous space, at $t = h$ requires

$$\frac{2}{h}\big(\boldsymbol{P}_2^w - \boldsymbol{P}_1^w\big) = \frac{2}{(1-h)}\big(\boldsymbol{P}_3^w - \boldsymbol{P}_2^w\big), \tag{A.3}$$

where $\boldsymbol{P}_i^w = w_i \boldsymbol{P}_i$, are the counterparts of $\boldsymbol{P}_i$ in homogeneous space. This results in

$$\begin{cases} w_2 \boldsymbol{P}_2 = h w_3 \boldsymbol{P}_3 + (1-h) w_1 \boldsymbol{P}_1, \\ w_2 = h w_3 + (1-h) w_1. \end{cases} \tag{A.4}$$

From (A.2) and (A.4), we have the following remarks:

- $\boldsymbol{C}(t)$ is generally C^0 continuous only even if (A.2a) is satisfied. It becomes C^1 continuous only under a specific parametrization, i.e., the interior knot h is specified as in (A.2b).
- The value of w_2 only affects the C^1 continuity for $\boldsymbol{C}^w(t)$ but not for $\boldsymbol{C}(t)$.
- The C^1 continuity of $\boldsymbol{C}^w(t)$ ensures the C^1 continuity of $\boldsymbol{C}(t)$ since satisfaction of (A.4) will guarantee the satisfaction of (A.2).
  The C^1 continuity of $\boldsymbol{C}(t)$ does not ensure the C^1 continuity of $\boldsymbol{C}^w(t)$ since satisfaction of (A.2) does not guarantee the satisfaction of (A.4). To achieve the C^1 continuity of $\boldsymbol{C}^w(t)$, the following additional condition is required:

$$w_2 = \frac{w_1 w_3}{\alpha w_1 + (1-\alpha) w_3}. \tag{A.5}$$

From (A.5) we can also see that having the same value for w_1, w_2 and w_3 is the easiest way (but not the necessary way) for $\boldsymbol{C}^w(t)$ to be C^1 continuous.

Table B.1

	Control points	Weights
$\boldsymbol{C}_0(t)$	$\{0, -1, 0\}$, $\{0, 0, 0\}$, $\{1, 0, 0\}$, $\{3, 0, 0\}$, $\{3, 1, 0\}$	$\{1, 2, 1, 1, 1\}$
$\boldsymbol{C}_1(t)$	$\{0, -1, 1.5\}$, $\{0, 0, 1.5\}$, $\{2.25, 0, 1.5\}$, $\{3, 0, 1.5\}$, $\{3, 1, 1.5\}$	$\{1, 1, 1, 3, 1\}$
$\boldsymbol{C}_2(t)$	$\{0, -1, 3\}$, $\{0, 0, 3\}$, $\{1.5, 0, 3\}$, $\{3, 0, 3\}$, $\{3, 1, 3\}$	$\{1, 1, 1, 1, 1\}$
$\boldsymbol{C}_3(t)$	$\{0, -1, 4.5\}$, $\{0, 0, 4.5\}$, $\{1, 0, 4.5\}$, $\{3, 0, 4.5\}$, $\{3, 1, 4.5\}$	$\{1, 4, 1, 2, 1\}$

Appendix B

The control points and the weights for the section curves used in producing Fig. 1 are listed in Table B.1 for readers who are interested to reproduce this example.

All four curves have the same knot sequence: $\{0, 0, 0, \frac{1}{2}, \frac{1}{2}, 1, 1, 1\}$.

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