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GEOMETRIC INTERPOLATION BY PLANAR CUBIC G^1 SPLINES

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Abstract.

In this paper, geometric interpolation by G^1 cubic spline is studied. A wide class of sufficient conditions that admit a G^1 cubic spline interpolant is determined. In particular, convex data as well as data with inflection points are included. The existence requirements are based upon geometric properties of data entirely, and can be easily verified in advance. The algorithm that carries out the verification is added.

AMS subject classification (2000): 65D05, 65D07, 65D17.

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1 Introduction.

Geometric interpolation schemes, introduced in [1], are becoming more and more important practical tool in the approximation of curves and surfaces. Perhaps the main reason could be found in the fact that such interpolants please the human eye more than usual linear counterparts. This is clearly a consequence of the basic principle of the geometric interpolation: free parameters of a parametric interpolant are determined by geometric conditions only. An interpolant should pass through a point, should have a prescribed tangent or normal direction, a curvature, etc. But, no additional artificial conditions are imposed on it such as at which parameter values the interpolation conditions should be met. Since no free parameters are used ineffectively, geometric interpolation often results in higher approximation order than one would expect from the functional case.

But geometric schemes involve a nonlinear part, and the questions like the existence and the efficient implementation require a more subtle analysis. Most of the results obtained are based upon the asymptotic analysis, and only a few papers examine geometric conditions on given data ([8], [6], [5]). A unified theory of geometric Hermite interpolation for parametric curves could be found in [2], and an excellent recent overview of the results is given in [3].

However, results offered by the asymptotic analysis are not always adequate in practical applications. If one is merely looking for an interpolant of a nice shape, suppositions like "if data points are sampled dense enough" are not very encouraging. Therefore robust algorithms should be based upon conditions that ensure the existence in advance if ever possible. But in geometric interpolation, this can be achieved very rarely. In this paper, we show that it can be done in the case of planar G^1 cubic spline interpolation. The interpolating problem concerned is the following. Let

$$\boldsymbol{T}_i \in \mathbb{R}^2, \quad i = 0, 1, 2, \dots, 3m, \quad \boldsymbol{T}_i \neq \boldsymbol{T}_{i+1}, \tag{1.1}$$

be a given sequence of data points. Find a cubic G^1 spline curve $\boldsymbol{P} : [a, b] \to \mathbb{R}^2$ with breakpoints

$$a := u_0 < u_1 < \cdots < u_m := b$$

that interpolates the data $\boldsymbol{T}_i$ in the prescribed order so that $\boldsymbol{P}(u_\ell) = \boldsymbol{T}_{3\ell}$. Let $\boldsymbol{d}_{3\ell}$, $\|\boldsymbol{d}_{3\ell}\|_2 = 1$, denote the tangent directions of the spline curve $\boldsymbol{P}$ at u_ℓ. A piecewise representation

$$\boldsymbol{P}^\ell\left(t^\ell\right) := \boldsymbol{P}(u)\Big|_{[u_{\ell-1}, u_\ell]}, \quad t^\ell := \frac{u - u_{\ell-1}}{\Delta u_{\ell-1}} \in [0, 1], \quad \ell = 1, 2, \dots, m,$$

rewrites the interpolation problem as follows: find cubic polynomials $\boldsymbol{P}^\ell$ such that

$$\begin{aligned} &\boldsymbol{P}^\ell\left(t_i^\ell\right) = \boldsymbol{T}_{3(\ell-1)+i}, \quad i = 0, \dots, 3, \\ &\frac{d}{dt^\ell}\boldsymbol{P}^\ell(0) = \alpha_0^\ell \boldsymbol{d}_{3(\ell-1)}, \quad \frac{d}{dt^\ell}\boldsymbol{P}^\ell(1) = \alpha_3^\ell \boldsymbol{d}_{3\ell}, \end{aligned} \qquad \ell = 1, 2, \dots, m, \tag{1.2}$$

where the unknown parameters t_1^ℓ, t_2^ℓ, $\alpha_0^\ell, \alpha_3^\ell$ must satisfy

$$0 =: t_0^\ell < t_1^\ell < t_2^\ell < t_3^\ell := 1, \quad \alpha_0^\ell > 0, \quad \alpha_3^\ell > 0, \quad \ell = 1, 2, \dots, m. \tag{1.3}$$

Here, Δ denotes the forward finite difference. Note that α_i^ℓ are chosen as local derivatives lengths rather than global in order to simplify the notation of further discussion. The tangent directions $\boldsymbol{d}_{3\ell}$, $\ell = 1, 2, \dots, m-1$, have clearly not been prescribed by the data (1.1) yet. However, they may be known as data or given as an approximation, perhaps as interactive shape parameters, or implicitly prescribed by the requirement that $\boldsymbol{P}$ is G^2 too. In the latter case, $\boldsymbol{d}_0$ and $\boldsymbol{d}_{3m}$ would be known, and the following $m-1$ equations

$$\begin{aligned} &\frac{1}{\left(\alpha_3^\ell\right)^2} \det\left(3\left(\boldsymbol{T}_{3\ell} - \boldsymbol{T}_{3\ell-3}\right) - \alpha_0^\ell \boldsymbol{d}_{3\ell-3},\, \boldsymbol{d}_{3\ell}\right) = \\ &\frac{1}{\left(\alpha_0^{\ell+1}\right)^2} \det\left(\boldsymbol{d}_{3\ell},\, 3\left(\boldsymbol{T}_{3\ell+3} - \boldsymbol{T}_{3\ell}\right) - \alpha_3^{\ell+1} \boldsymbol{d}_{3\ell+3}\right), \end{aligned} \qquad \ell = 1, 2, \dots, m-1, \tag{1.4}$$

added. But, in general, the problem (1.2) and (1.3) need not have a solution. So it is quite possible that the curve $\boldsymbol{P}$ could not interpolate all the prescribed data. For this reason we split the interpolation problem (1.2) and (1.3) into two steps. At the first and the main step, we determine the region for $(\boldsymbol{d}_\ell)_{\ell=1}^m$ that admits a solution of (1.2). The second step is left to the user, but with clear

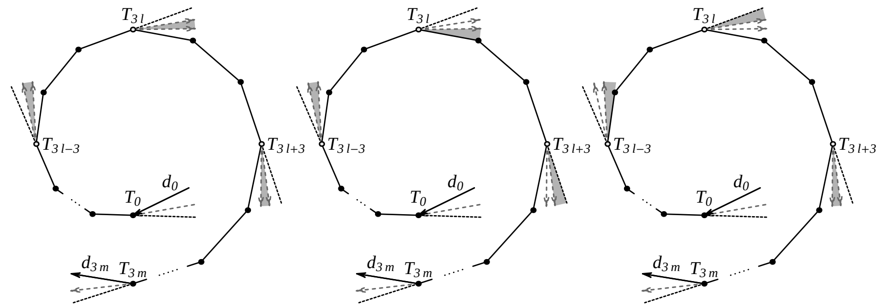

Figure 1.1: The directions for tangents (gray area) that imply the existence of a G^1 spline $\boldsymbol{P}$ for convex data.

bounds on $\boldsymbol{d}_{3\ell}$. Some suggestions how to choose tangent directions are given in Section 4.

As expected, it is not possible to break apart sufficient conditions that admit a solution to a local level. However, if the data are convex, we are able to determine possible angles that $\boldsymbol{d}_{3\ell}$ is allowed to take by the local data only. To be precise, at a point $\boldsymbol{T}_{3\ell}$ the angle between $\Delta\boldsymbol{T}_{3\ell-1}$ and $\Delta\boldsymbol{T}_{3\ell}$ gives a range for $\boldsymbol{d}_{3\ell}$ that is further split into at most three subangles. This partition depends only on data $\boldsymbol{T}_{3\ell-3}, \boldsymbol{T}_{3\ell-2}, \ldots, \boldsymbol{T}_{3\ell+3}$. All that is left is to connect particular subangles in an allowable global choice by taking into account certain simple additional relations between subangles at different breakpoints. This is carried out by a straightforward backtracking algorithm. Figure 1.1 shows three such possible choices (grayed). But if the data imply an inflexion point, the answer is not so obvious, and is left to Section 4, as well as the precise explanation of the convex case.

The outline of the paper is the following. In Section 2 a polynomial case is considered and geometric conditions that imply the existence of the interpolant are derived. Section 3 is devoted to a proof of two main theorems of Section 2. In Section 4 the results are carried over to G^1 cubic spline curves, and the conclusions are presented as an algorithm.

2 Polynomial case.

The first step to the G^1 spline construction is a single polynomial case. So, $m = 1$, and $\boldsymbol{P}^1 = \boldsymbol{P}$. Further, let us shorten the notation by

$$\boldsymbol{d}_0 := \boldsymbol{d}_0^1, \quad \boldsymbol{d}_3 := \boldsymbol{d}_3^1, \quad t_1 := t_1^1, \quad t_2 := t_2^1, \quad \alpha_0 := \alpha_0^1, \quad \alpha_3 := \alpha_3^1.$$

The nonlinear part of the interpolation problem (1.2) is to compute the admissible parameters $(t_1, t_2, \alpha_0, \alpha_3) \in \mathcal{D}$, where by (1.3)

$$\mathcal{D} := \{(t_1, t_2);\ 0 =: t_0 < t_1 < t_2 < t_3 := 1\} \times \{(\alpha_0, \alpha_3);\ \alpha_0 > 0,\ \alpha_3 > 0\},$$

is an open set with the boundary $\partial\mathcal{D}$, determined by $t_i = t_{i+1}$ for at least one $i \in \{0, 1, 2\}$, $\alpha_0 = 0$ or $\alpha_3 = 0$. Once this parameters are determined, the coefficients of $\boldsymbol{P}$ are obtained by using any standard interpolation scheme componentwise. The problem of finding geometric conditions is very close to the problem considered in [6], where the interpolation of six points by a cubic polynomial curve is considered.

To reduce the interpolation problem (1.2) to the nonlinear system for unknown $(t_1, t_2, \alpha_0, \alpha_3)$ only, divided differences that map polynomials of degree ≤ 3 to zero are applied to (1.2). Therefore,

$$[t_0, t_0, t_1, t_2, t_3]\boldsymbol{P} = \boldsymbol{0} = \frac{\alpha_0}{\dot{\omega}(t_0)}\boldsymbol{d}_0 + \sum_{j=1}^{3}\left(\sum_{i=j}^{3}\frac{1}{\dot{\omega}(t_i)}\frac{1}{t_i - t_0}\right)\Delta\boldsymbol{T}_{j-1}, \tag{2.1}$$

$$[t_0, t_1, t_2, t_3, t_3]\boldsymbol{P} = \boldsymbol{0} = \frac{\alpha_3}{\dot{\omega}(t_3)}\boldsymbol{d}_3 + \sum_{j=0}^{2}\left(\sum_{i=0}^{j}\frac{1}{\dot{\omega}(t_i)}\frac{1}{t_3 - t_i}\right)\Delta\boldsymbol{T}_{j}, \tag{2.2}$$

where

$$\omega(t) := \prod_{i=0}^{3}(t - t_i).$$

Further, with linear functionals $\det(\cdot, \Delta\boldsymbol{T}_0)$, $\det(\cdot, \Delta\boldsymbol{T}_1)$ applied to (2.1), and $\det(\cdot, \Delta\boldsymbol{T}_1)$, $\det(\cdot, \Delta\boldsymbol{T}_2)$ applied to (2.2) one obtains

$$\begin{aligned}
&\frac{\alpha_0}{\dot{\omega}(t_0)}\det(\boldsymbol{d}_0, \Delta\boldsymbol{T}_k) + \sum_{j=1}^{3}\left(\sum_{i=j}^{3}\frac{1}{\dot{\omega}(t_i)}\frac{1}{t_i - t_0}\right)\det(\Delta\boldsymbol{T}_{j-1}, \Delta\boldsymbol{T}_k) = 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 0, 1.\\
&\frac{\alpha_3}{\dot{\omega}(t_3)}\det(\boldsymbol{d}_3, \Delta\boldsymbol{T}_{k+1}) + \sum_{j=0}^{2}\left(\sum_{i=0}^{j}\frac{1}{\dot{\omega}(t_i)}\frac{1}{t_3 - t_i}\right)\det(\Delta\boldsymbol{T}_j, \Delta\boldsymbol{T}_{k+1}) = 0,
\end{aligned} \tag{2.3}$$

Let us recall that $t_0 = 0$ and $t_3 = 1$. After eliminating α_0 from the first and α_3 from the last equation, the system transforms to

$$\begin{aligned}
&\frac{1}{t_1^2(1-t_1)} - \frac{1}{t_2^2(1-t_2)}(1+\mu_1) + \frac{t_2 - t_1}{(1-t_1)(1-t_2)}\left(1 + \mu_1(1+\lambda_1) - \frac{\lambda_1}{\lambda_2}\right) = 0,\\
&\frac{1}{t_2(1-t_2)^2} - \frac{1}{t_1(1-t_1)^2}(1+\mu_2) + \frac{t_2 - t_1}{t_1 t_2}\left(1 + \mu_2(1+\lambda_2) - \frac{\lambda_2}{\lambda_1}\right) = 0,
\end{aligned} \tag{2.4}$$

and

$$\alpha_0 = \delta_1\frac{t_1 t_2}{t_2 - t_1}\left(\frac{1}{t_2^2(1-t_2)} - \frac{t_2 - t_1}{(1-t_1)(1-t_2)}(1+\lambda_1)\right), \tag{2.5}$$

$$\alpha_3 = \delta_2\frac{(1-t_1)(1-t_2)}{t_2 - t_1}\left(\frac{1}{t_1(1-t_1)^2} - \frac{t_2 - t_1}{t_1 t_2}(1+\lambda_2)\right), \tag{2.6}$$

where the new constants introduced are defined more generally as

$$
\begin{aligned}
&D_{i,j} := \det(\Delta \boldsymbol{T}_i, \Delta \boldsymbol{T}_j),\\
&\lambda_{2\ell-1} := \frac{D_{3\ell-3,3\ell-1}}{D_{3\ell-3,3\ell-2}}, \qquad \lambda_{2\ell} := \frac{D_{3\ell-3,3\ell-1}}{D_{3\ell-2,3\ell-1}},\\
&\mu_{2\ell-1} := \frac{\det(\boldsymbol{d}_{3\ell-3}, \Delta \boldsymbol{T}_{3\ell-2})}{\det(\boldsymbol{d}_{3\ell-3}, \Delta \boldsymbol{T}_{3\ell-3})}, \quad \mu_{2\ell} := \frac{\det(\Delta \boldsymbol{T}_{3\ell-2}, \boldsymbol{d}_{3\ell})}{\det(\Delta \boldsymbol{T}_{3\ell-1}, \boldsymbol{d}_{3\ell})},\\
&\delta_{2\ell-1} := \frac{D_{3\ell-3,3\ell-2}}{\det(\boldsymbol{d}_{3\ell-3}, \Delta \boldsymbol{T}_{3\ell-3})}, \quad \delta_{2\ell} := \frac{D_{3\ell-2,3\ell-1}}{\det(\Delta \boldsymbol{T}_{3\ell-1}, \boldsymbol{d}_{3\ell})}.
\end{aligned}
\tag{2.7}
$$

They have a clear geometric meaning, for example, $D_{i,j}$ is the volume of a

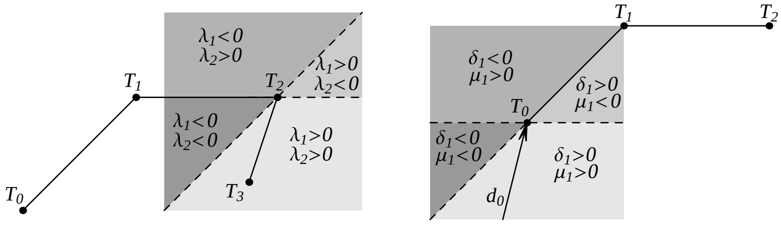

Figure 2.1: The signs of λ_1, λ_2 depending on the position of $\boldsymbol{T}_3$ (left), and the signs of μ_1, δ_1 depending on the tangent direction $\boldsymbol{d}_0$ (right).

parallelogram spanned by vectors $\Delta \boldsymbol{T}_i$, $\Delta \boldsymbol{T}_j$ and the other constants are the ratios of such volumes. Fig 2.1 illustrates the sign change in λ_1 and λ_2 (μ_1 and δ_1) as $\boldsymbol{T}_3$ (tangent direction $\boldsymbol{d}_0$) changes. Note also that $\lambda_{2\ell-1}, \lambda_{2\ell}$ depend on data points $\boldsymbol{T}_i$ only. Further, for the future use, we add the following observation.

REMARK 2.1. *The constants* $\mu_{2\ell-1}, \mu_{2\ell}$, *and* $\operatorname{sign} \delta_{2\ell-1}, \operatorname{sign} \delta_{2\ell}$ *do not depend on the length of tangents involved.*

REMARK 2.2. *The system of equations* (2.3) *could also be derived from [6, eq.(4)] by replacing* $\boldsymbol{T}_1$, $\boldsymbol{T}_4$ *with* $\boldsymbol{T}_0 + (t_1 - t_0)\alpha_0 \boldsymbol{d}_0$, $\boldsymbol{T}_5 - (t_5 - t_4)\alpha_3 \boldsymbol{d}_3$ *accordingly, and passing to the limits* $t_1 \to t_0$ *and* $t_4 \to t_5$ *as well as renumbering the remaining points* $\boldsymbol{T}_i$ *and parameters* t_i, $i = 0, 2, 3, 5$, *by* $0, 1, 2, 3$. *Some of the properties of the nonlinear system* (2.3) *are thus inherited from [6, eq.(8)–(11)], but not all. In particular, the requirement* $\alpha_i > 0$ *has to be considered thoroughly.*

In order to make the analysis bearable some restrictions on the data must be made. Namely, $\lambda_k > 0$, $\mu_k > 0$ and $\delta_k > 0$, $k = 1, 2$, will be assumed for the convex data and $\lambda_1 \cdot \lambda_2 < 0$, $\delta_k > 0$ for the data that imply an inflection point. Since the individual pieces will be composed in a spline curve, these assumptions are very natural as one can see from Fig 2.1.

It is straightforward to compute the solution of the system (2.4) in a closed form by using Gröbner basis or resultants. But not all the solutions will satisfy $0 < t_1 < t_2 < 1$. Even if this is true, the solution may not produce positive α_0 and α_3. This means that we are dealing with a problem that is only partly algebraic. The following lemmas reveal the possibility that $\boldsymbol{P}'$ vanishes at the boundary.

LEMMA 2.1. *Suppose that* $\lambda_1 > 0$. *There exists a unique solution of the system* (2.4) *and* (2.5) *such that* $0 < t_1 < t_2 < 1$ *and* $\alpha_0 = 0$ *if and only if* $\lambda_2 > 0$ *and* $\mu_2 = \phi_2(\lambda_1, \lambda_2)$, *where*

$$\phi_2(\lambda_1,\lambda_2) := \frac{\lambda_2 \dfrac{1-\tilde{t}_1}{\tilde{t}_2^2} - \lambda_1 \dfrac{\tilde{t}_1}{(1-\tilde{t}_2)^2}}{\lambda_2 \dfrac{1-\tilde{t}_2}{\tilde{t}_1^2} - \lambda_1 \dfrac{\tilde{t}_2}{(1-\tilde{t}_1)^2}} - 1,$$

and $\left(\tilde{t}_1, \tilde{t}_2\right)$ *is the unique solution of the system*

$$\frac{1-t_1}{t_2^2(t_2-t_1)} = 1+\lambda_1, \qquad \frac{1-t_2}{t_1^2(t_2-t_1)} = \frac{\lambda_1}{\lambda_2}\left(1+\lambda_2\right), \qquad 0 < t_1 < t_2 < 1. \tag{2.8}$$

LEMMA 2.2. *Suppose that* $\lambda_1 > 0$. *There exists a unique solution of the system* (2.4) *and* (2.6) *such that* $0 < t_1 < t_2 < 1$ *and* $\alpha_3 = 0$ *if and only if* $\lambda_2 > 0$ *and* $\mu_1 = \phi_1(\lambda_1, \lambda_2) := \phi_2(\lambda_2, \lambda_1)$.

PROOF. Let us prove Lemma 2.1. The proof of Lemma 2.2 is similar and will be omitted. When $\alpha_0 = 0$ the equations (2.4) and (2.5) simplify to (2.8) and

$$\mu_2 = \frac{\lambda_2 \dfrac{1-t_1}{t_2^2} - \lambda_1 \dfrac{t_1}{(1-t_2)^2}}{\lambda_2 \dfrac{1-t_2}{t_1^2} - \lambda_1 \dfrac{t_2}{(1-t_1)^2}} - 1.$$

From the first equation in (2.8), one obtains

$$t_1(t_2) = \frac{(1+\lambda_1)t_2^3 - 1}{(1+\lambda_1)t_2^2 - 1}.$$

Since $\lambda_1 > 0$, function $t_1(t_2)$ has only one real zero $t_2 = \dfrac{1}{\sqrt[3]{1+\lambda_1}}$ and one positive real pole $t_2 = \dfrac{1}{\sqrt{1+\lambda_1}}$, where

$$0 < \frac{1}{\sqrt{1+\lambda_1}} < \frac{1}{\sqrt[3]{1+\lambda_1}} < 1.$$

Moreover, $t_1(0) = t_1(1) = 1$, $t_1(t_2) = t_2$ iff $t_2 = 1$, and $t_1(t_2)$ is monotonically increasing. Namely,

$$\frac{d}{dt_2} t_1(t_2) = \frac{(1+\lambda_1)t_2(2 - 3t_2 + (1+\lambda_1)t_2^3)}{((1+\lambda_1)t_2^2 - 1)^2} > 0, \quad t_2 \in (0,1].$$

The condition $0 < t_1(t_2) < t_2 < 1$ is thus fulfilled iff $t_2 \in \left(\frac{1}{\sqrt[3]{1+\lambda_1}}, 1\right)$. By substituting $t_1(t_2)$ into the second equation in (2.8) it simplifies to

$$(t_2 - 1)g(t_2) = 0, \quad g(t_2) := \lambda_2 - \frac{\lambda_1(1+\lambda_2)((1+\lambda_1)t_2^3 - 1)^2}{((1+\lambda_1)t_2^2 - 1)^3}.$$

Now,

$$g\left(\frac{1}{\sqrt[3]{1+\lambda_1}}\right) = \lambda_2, \quad g(1) = -1,$$

and the sign of the derivative

$$\frac{d}{dt_2} g(t_2) = \frac{6\lambda_1(1+\lambda_1)(1+\lambda_2)t_2(t_2-1)((1+\lambda_1)t_2^3 - 1)}{((1+\lambda_1)t_2^2 - 1)^4}$$

is equal to the sign of $1 + \lambda_2$ for $t_2 \in \left(\frac{1}{\sqrt[3]{1+\lambda_1}}, 1\right)$. Therefore a unique $\tilde{t}_2 \in \left(\frac{1}{\sqrt[3]{1+\lambda_1}}, 1\right)$ that solves $g(\tilde{t}_2) = 0$ exists iff $\lambda_2 > 0$. Then $\left(\tilde{t}_1, \tilde{t}_2\right) := \left(t_1(\tilde{t}_2), \tilde{t}_2\right)$ is the unique solution of the system (2.8), which concludes the proof. □

Let us now define two additional functions that will play a major role in the formulation of main results, namely

$$\phi_3(\lambda_1, \lambda_2, \mu_1) := \frac{\lambda_2 \mu_1}{\lambda_1(\lambda_2\mu_1 - 1 - \sqrt{1+\mu_1})},$$
$$\phi_4(\lambda_1, \lambda_2, \mu_1) := \frac{\lambda_2\mu_1(\lambda_2\mu_1(1+2\lambda_1) - 2\lambda_1)}{\lambda_1^2(\lambda_2\mu_1 - 1)^2}.$$

The next lemma collects some of their properties that can easily be verified.

LEMMA 2.3. *Suppose that $\lambda_1 > 0$, $\lambda_2 < 0$ and $\mu_1 > 0$. Then $\phi_3(\lambda_1, \lambda_2, \cdot)$ and $\phi_4(\lambda_1, \lambda_2, \cdot)$ are monotonically increasing functions of μ_1,*

$$\lim_{\mu_1 \to \infty} \phi_3(\lambda_1, \lambda_2, \mu_1) = \frac{1}{\lambda_1}, \quad \lim_{\mu_1 \to \infty} \phi_4(\lambda_1, \lambda_2, \mu_1) = \frac{1+2\lambda_1}{\lambda_1^2},$$

and $\phi_3(\lambda_1, \lambda_2, \cdot) < \phi_4(\lambda_1, \lambda_2, \cdot)$. Moreover $\phi_3(\lambda_1, \lambda_2, \mu_1) = \mu_2$ if and only if $\phi_4(\lambda_2, \lambda_1, \mu_2) = \mu_1$, and $\phi_4(\lambda_1, \lambda_2, \mu_1) = \mu_2$ if and only if $\phi_3(\lambda_2, \lambda_1, \mu_2) = \mu_1$.

The following results now give sufficient conditions on data points and tangent directions that imply the existence of the interpolant $\boldsymbol{P}$. The first assertion covers convex data, and the second one covers data with an inflection point.

THEOREM 2.4. *Suppose that the data $\boldsymbol{d}_0, \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3, \boldsymbol{d}_3$ satisfy*

$$\lambda_k > 0, \quad \delta_k > 0, \quad \mu_k > 0, \quad k = 1, 2.$$

If

$$0 < \mu_1 < \phi_1(\lambda_1, \lambda_2) \quad \text{and} \quad 0 < \mu_2 < \phi_2(\lambda_1, \lambda_2),$$

or

$$\mu_1 > \phi_1(\lambda_1, \lambda_2) \quad \textit{and} \quad \mu_2 > \phi_2(\lambda_1, \lambda_2),$$

then the interpolating curve $\boldsymbol{P}$ *that satisfies* (1.2) *exists.*

THEOREM 2.5. *Suppose that the data* $\boldsymbol{d}_0, \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3, \boldsymbol{d}_3$ *satisfy*

$$\lambda_1 > 0, \quad \lambda_2 < 0, \quad \delta_1 > 0 \quad \textit{and} \quad \delta_2 > 0.$$

If $\mu_1 > 0$ *and*

$$\phi_3(\lambda_1, \lambda_2, \mu_1) < \mu_2 < \phi_4(\lambda_1, \lambda_2, \mu_1),$$

then the interpolating curve $\boldsymbol{P}$ *that satisfies* (1.2) *exists.*

REMARK 2.3. *The symmetry of equations* (2.4)-(2.6) *implies that Theorem 2.5 holds also if the role of* λ_1, λ_2, *and* μ_1, μ_2 *is reversed.*

3 Proof of Theorem 2.4 and Theorem 2.5

The key part of the proof is the following observation.

LEMMA 3.1. *Suppose that the assumptions of Theorem 2.4 or Theorem 2.5 are met. Then the system* (2.4)–(2.6) *cannot have a solution arbitrary close to the boundary* $\partial\mathcal{D}$.

PROOF. By passing to the limit mentioned by Remark 2.2 in the discussion [6, Section 4] one can check that only two possible cases $\Delta t_i \to 0$ are to be considered. The first one, $\Delta t_0 \to 0$ and $\Delta t_1 \to 0$, implies $\lambda_1\lambda_2 < 0$ and $\lambda_1 \to \frac{1}{\mu_2} + \frac{\lambda_1(1+\sqrt{1+\mu_1})}{\lambda_2\mu_1}$ or equivalently $\mu_2 \to \phi_3(\lambda_1, \lambda_2, \mu_1)$. Similarly, the second one, $\Delta t_1 \to 0$ and $\Delta t_2 \to 0$, implies $\lambda_1\lambda_2 < 0$ too, and $\lambda_1 \to \frac{\lambda_1}{\lambda_2\mu_1} + \frac{1+\sqrt{1+\mu_2}}{\mu_2}$ or equivalently $\mu_2 \to \phi_4(\lambda_1, \lambda_2, \mu_1)$. The assertion follows now from Lemma 2.1 and Lemma 2.2. □

A standard degree type argument will now conclude the proofs. Let us first show that the number of admissible solutions for the particular data that satisfy the conditions of theorems is odd. The data points are chosen as

$$\boldsymbol{T}_0 = \begin{pmatrix} -4 \\ -4 \end{pmatrix}, \quad \boldsymbol{T}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{T}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad \boldsymbol{T}_3 = \begin{pmatrix} 9+(-1)^s \\ (-1)^s 4 \end{pmatrix}, \quad s \in \{0, 1\},$$

where $s = 1$ corresponds to the convex case and $s = 0$ to the other one. Further, the tangent directions are chosen as

data 1: $\boldsymbol{d}_0 = (2,3)^T, \quad d_3 = (2,-3)^T, \quad s = 1,$

data 2: $\boldsymbol{d}_0 = (-2,2)^T, \quad d_3 = (-2,-2)^T, \quad s = 1,$

data 3: $\boldsymbol{d}_0 = (-2,2)^T, \quad d_3 = (-1,2)^T, \quad s = 0.$

Table 3.1: The constants for the particular data.

	λ_1	λ_2	μ_1	μ_2	δ_1	δ_2
data 1	2	2	3	3	4	4
data 2	2	2	$\frac{1}{2}$	$\frac{1}{2}$	1	1
data 3	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1

Table 3.2: The admissible solutions for the particular data.

	t_1	t_2	α_0	α_3	multiplicity
data 1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	3
data 2	$\frac{1}{3}(3-\sqrt{3})$	$\frac{\sqrt{3}}{3}$	$3(1+\sqrt{3})$	$3(1+\sqrt{3})$	1
data 3	0.450047	0.583425	12.1642	12.1828	1

Table 3.1 shows the values of the constants (2.7) and Table 3.2 gives the corresponding admissible solutions in $\mathcal{D}$. Since $\phi_1(\lambda_1,\lambda_2)=\phi_2(\lambda_1,\lambda_2)=2.80828$ for data 1 and data 2, and

$$\phi_3(\lambda_1,\lambda_2,\mu_1)=10-4\sqrt{6}<\mu_2=\frac{1}{2}<\phi_4(\lambda_1,\lambda_2,\mu_1)=\frac{24}{25}$$

for data 3, the assumptions of theorems are fulfilled.

A homotopy will now be used to carry the conclusions from the particular case outlined in Table 3.1 to the general one. Let us denote the system (2.4)–(2.6) by $\boldsymbol{F}(\boldsymbol{t},\boldsymbol{\alpha};\boldsymbol{\lambda},\boldsymbol{\delta},\boldsymbol{\mu})=\boldsymbol{0}$, where

$$\boldsymbol{t}=(t_1,t_2),\quad \boldsymbol{\alpha}=(\alpha_1,\alpha_2),\quad \boldsymbol{\lambda}=(\lambda_1,\lambda_2),\quad \boldsymbol{\delta}=(\delta_1,\delta_2),\quad \boldsymbol{\mu}=(\mu_1,\mu_2).$$

Further, let $(\boldsymbol{\lambda},\boldsymbol{\delta},\boldsymbol{\mu})$ stand for general data, and $(\boldsymbol{\lambda}^*,\boldsymbol{\delta}^*,\boldsymbol{\mu}^*)$ for the particular case. A homotopy is chosen as $\boldsymbol{H}(\boldsymbol{t},\boldsymbol{\alpha};\zeta):=\boldsymbol{F}\left(\boldsymbol{t},\boldsymbol{\alpha};\boldsymbol{\lambda}(\zeta),\boldsymbol{\delta}(\zeta),\boldsymbol{\mu}(\zeta)\right)$, where

$$\boldsymbol{\lambda}(\zeta):=(1-\zeta)\boldsymbol{\lambda}^*+\zeta\boldsymbol{\lambda},\quad \boldsymbol{\delta}(\zeta):=(1-\zeta)\boldsymbol{\delta}^*+\zeta\boldsymbol{\delta},$$

and $\mu_k:[0,1]\to\mathbb{R}$ satisfies $\mu_k(0)=\mu_k^*$, $\mu_k(1)=\mu_k$ for $k=1,2$. As is [6, Section 6] it is straightforward to see that $\mu_k(\cdot)$, $k=1,2$, can be chosen as continuous piecewise linear functions so that the data with constants $\boldsymbol{\lambda}(\zeta)$, $\boldsymbol{\delta}(\zeta)$ and $\boldsymbol{\mu}(\zeta)$ meet the requirements of Theorem 2.4 or Theorem 2.5 for any $\zeta\in[0,1]$. Therefore, by Lemma 3.1, there exists a compact set $K\subset\mathcal{D}$ so that

$$\mathcal{S}:=\{(\boldsymbol{t},\boldsymbol{\alpha})\in\mathcal{D};\quad \boldsymbol{H}(\boldsymbol{t},\boldsymbol{\alpha};\zeta)=\boldsymbol{0},\quad \zeta\in[0,1]\}\subset K,\quad \mathcal{S}\cap\partial K=\emptyset.$$

A Brouwer's degree ([7]) of $\boldsymbol{H}$ on K is thus invariant for all $\zeta\in[0,1]$. But since it is odd for the particular map $\boldsymbol{F}(\cdot,\cdot;\boldsymbol{\lambda}^*,\boldsymbol{\delta}^*,\boldsymbol{\mu}^*)$, equations $\boldsymbol{F}(\boldsymbol{t},\boldsymbol{\alpha};\boldsymbol{\lambda},\boldsymbol{\delta},\boldsymbol{\mu})=\boldsymbol{0}$ must have at least one admissible solution and Theorem 2.4 and Theorem 2.5 are proved.

4 The G^1 spline curve

We tackle now the G^1 cubic spline interpolation as introduced in Section 1, with tangent directions in (1.2) considered to be unknown. Each tangent direction $\boldsymbol{d}_{3\ell}$ depends on one parameter only. If vectors $\Delta\boldsymbol{T}_{3\ell-1}$ and $\Delta\boldsymbol{T}_{3\ell}$ are not collinear, i.e., $D_{3\ell-1,3\ell} \neq 0$, we may express them as

$$\begin{aligned} &\boldsymbol{d}_0 := \boldsymbol{d}_0(\xi_0) := (\xi_0 - 1)\Delta\boldsymbol{T}_1 + \xi_0\Delta\boldsymbol{T}_0, \\ &\boldsymbol{d}_{3\ell} := \boldsymbol{d}_{3\ell}(\xi_\ell) := \sigma_{3\ell}(1-\xi_\ell)\Delta\boldsymbol{T}_{3\ell-1} + \sigma_{3\ell-1}\xi_\ell\Delta\boldsymbol{T}_{3\ell}, \quad \ell = 1,\ldots,m-1, \\ &\boldsymbol{d}_{3m} := \boldsymbol{d}_m(\xi_m) := (1-\xi_m)\Delta\boldsymbol{T}_{3m-1} - \xi_m\Delta\boldsymbol{T}_{3m-2}, \end{aligned} \tag{4.1}$$

with

$$\sigma_k := \operatorname{sign}\left(\frac{D_{k-1,k}}{D_{k,k+1}}\right).$$

The tangents introduced in (4.1) are not normalized, but by Remark 2.1 this is not important. Further, the definition (4.1) implies that some constants defined in (2.7) become explicit functions of ξ_ℓ. In particular,

$$\begin{aligned} \delta_{2\ell-1} = \delta_{2\ell-1}(\xi_{\ell-1}) &= \frac{1}{1-\xi_{\ell-1}}\left|\frac{D_{3\ell-3,3\ell-2}}{D_{3\ell-4,3\ell-3}}\right|, \quad \ell = 2,3,\ldots,m, \\ \delta_{2\ell} = \delta_{2\ell}(\xi_\ell) &= \frac{1}{\xi_\ell}\left|\frac{D_{3\ell-2,3\ell-1}}{D_{3\ell-1,3\ell}}\right|, \quad \ell = 1,2,\ldots,m-1, \\ \delta_1 = \delta_1(\xi_0) = \frac{1}{1-\xi_0}, &\quad \delta_{2m} = \delta_{2m}(\xi_m) = \frac{1}{\xi_m}, \end{aligned} \tag{4.2}$$

shows that a requirement $\delta_{2\ell}(\xi_\ell) > 0$, $\delta_{2\ell+1}(\xi_\ell) > 0$ pins down ξ_ℓ to $(0,1)$ as can be seen in Figure 4.1. The constants $\mu_{2\ell-1}$, $\mu_{2\ell}$ turn out as

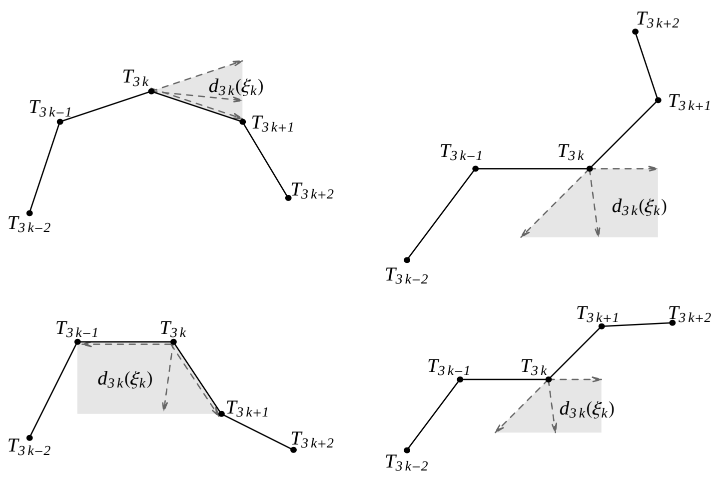

Figure 4.1: The tangent directions $\boldsymbol{d}_{3k}(\xi_k)$ for $\xi_k \in (0,1)$ (gray area).

$$\mu_{2\ell-1} = \mu_{2\ell-1}(\xi_{\ell-1}) = \sigma_{3\ell-4}\xi_{\ell-1}\delta_{2\ell-1}(\xi_{\ell-1}) + \frac{D_{3\ell-4,3\ell-2}}{D_{3\ell-4,3\ell-3}}, \quad \ell = 2,3,\ldots,m,$$

$$\mu_{2\ell} = \mu_{2\ell}(\xi_\ell) = \sigma_{3\ell}\,(1-\xi_\ell)\,\delta_{2\ell}(\xi_\ell) + \frac{D_{3\ell-2,3\ell}}{D_{3\ell-1,3\ell}}, \quad \ell = 1,2,\ldots,m-1, \tag{4.3}$$

$$\mu_1 = \mu_1(\xi_0) = \frac{\xi_0}{1-\xi_0}, \quad \mu_{2m} = \mu_{2m}(\xi_m) = \frac{1-\xi_m}{\xi_m}.$$

In view of Theorem 2.4 or Theorem 2.5 it is necessary to determine for which $\xi_\ell \in (0,1)$ the functions $\mu_{2\ell}$ and $\mu_{2\ell+1}$ are both positive. Let us recall the notation $f(\mathcal{I}) := \{f(x);\ x \in \mathcal{I}\}$, $f^{-1}(\mathcal{I}) := \{x;\ f(x) \in \mathcal{I}\}$. Then

$$\begin{aligned}
\mathcal{I}_0 &:= \mu_1^{-1}((0,\infty)) \cap (0,1) = (0,1),\\
\mathcal{I}_\ell &:= \mu_{2\ell}^{-1}((0,\infty)) \cap \mu_{2\ell+1}^{-1}((0,\infty)) \cap (0,1), \quad \ell = 1,2,\ldots,m-1,\\
\mathcal{I}_m &:= \mu_{2m}^{-1}((0,\infty)) \cap (0,1) = (0,1),
\end{aligned}$$

are the required subintervals, with $\mathcal{I}_\ell \neq \emptyset$ still to be assured. Let us restrict ourselves to the interval $(0,1)$ only. It is easy to see that $\mu_{2\ell}$ and $\mu_{2\ell+1}$ are both monotone as functions of ξ_ℓ. Moreover,

$$\begin{aligned}
&\lim_{\xi\downarrow 0}\mu_{2\ell}(\xi_\ell) = \sigma_{3\ell}\,\infty, \quad \mu_{2\ell}(1) = \frac{D_{3\ell-2,3\ell}}{D_{3\ell-1,3\ell}},\\
&\mu_{2\ell+1}(0) = \frac{D_{3\ell-1,3\ell+1}}{D_{3\ell-1,3\ell}}, \quad \lim_{\xi\uparrow 1}\mu_{2\ell+1}(\xi_\ell) = \sigma_{3\ell-1}\,\infty.
\end{aligned}$$

Therefrom it is easy to see that $\mu_{2\ell}^{-1}((0,\infty)) \cap (0,1) = \emptyset$ iff

$$\sigma_{3\ell} = -1, \quad D_{3\ell-2,3\ell}D_{3\ell-1,3\ell} \le 0, \tag{4.4}$$

and $\mu_{2\ell+1}^{-1}((0,\infty)) \cap (0,1) = \emptyset$ iff

$$\sigma_{3\ell-1} = -1, \quad D_{3\ell-1,3\ell+1}D_{3\ell-1,3\ell} \le 0. \tag{4.5}$$

Now, if conditions (4.4) and (4.5) are not fulfilled, each of the above intervals is nonempty, but that does not imply the intersection to be nonempty too. In this case it is easy to check that $\mu_{2\ell}^{-1}((0,\infty)) \cap \mu_{2\ell+1}^{-1}((0,\infty)) \cap (0,1) = \emptyset$ if and only if

$$\begin{aligned}
&D_{3\ell-2,3\ell-1}D_{3\ell,3\ell+1} > 0, \quad D_{3\ell-2,3\ell}D_{3\ell-1,3\ell+1} > 0,\\
&D_{3\ell-2,3\ell}D_{3\ell,3\ell+1} < 0, \quad \sigma_{3\ell}\,\mu_{2\ell}^{-1}(0) \le \sigma_{3\ell}\,\mu_{2\ell+1}^{-1}(0).
\end{aligned} \tag{4.6}$$

Let us summarize this discussion in the following theorem.

THEOREM 4.1. *Suppose that data points* (1.1) *satisfy*

$$\begin{aligned}
&\lambda_{2\ell-1} > 0,\ \lambda_{2\ell} > 0 \ \text{ or } \ \lambda_{2\ell-1}\lambda_{2\ell} < 0, \quad \ell = 1,2,\ldots,m,\\
&D_{3\ell-1,3\ell} \neq 0, \quad \ell = 1,2,\ldots,m-1,
\end{aligned}$$

and additionally none of the relations (4.4), (4.5) *or* (4.6) *is fulfilled. Further, let the tangents be given by* (4.1), *and the rest of the constants determined by* (4.2) *and* (4.3). *Then for every* $\xi_\ell \in \mathcal{I}_\ell$, $\ell = 0, 1, \ldots, m$, *the suppositions of either Theorem 2.4 or Theorem 2.5 are fulfilled on* ℓ*-th segment. Further, the algorithm* ForwardSweep *determines the admissible intervals for parameters* ξ_ℓ.

Only the algorithm is left to be constructed. We choose it to be a simple backtracking procedure that traverses the data (1.1) in a forward sweep $\boldsymbol{T}_0 \rightarrow \boldsymbol{T}_{3m}$ and determines an intermediate result

$$\Xi_\ell \subset \mathcal{I}_\ell, \quad \ell = 0, 1, \ldots, m,$$

in such a way that for any $\xi_\ell \in \Xi_\ell$ there exists a choice

$$\xi_i \in \Xi_i, \quad i = 0, 1, \ldots, \ell - 1,$$

such that $(\xi_0, \xi_1, \ldots, \xi_\ell)$ is admissible as far as data $\boldsymbol{T}_i$, $i = 0, 1, \ldots, 3\ell$, are concerned. A backward sweep $\boldsymbol{T}_{3m} \rightarrow \boldsymbol{T}_0$ shrinks the temporary Ξ_ℓ, $\ell = m - 1, m - 2, \ldots, 0$ so that for any $\xi_\ell \in \Xi_\ell$ there exists a choice

$$\xi_i \in \Xi_i, \quad i = 0, 1, \ldots, \ell - 1, \ell + 1, \ldots, m,$$

such that $(\xi_0, \xi_1, \ldots, \xi_m)$ is admissible for all data. The induction step $\Xi_{\ell-1} \rightarrow$

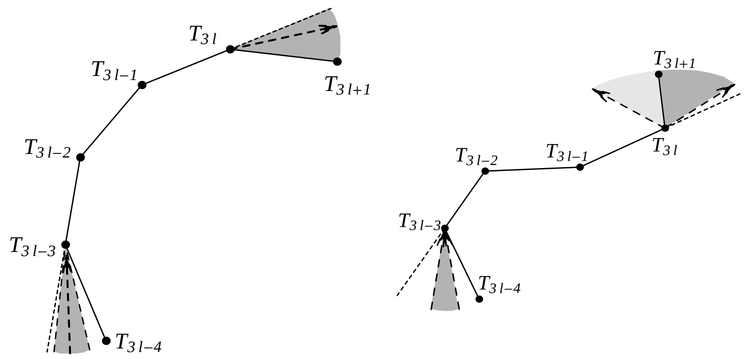

Figure 4.2: Induction step: $\lambda_{2\ell-1} > 0, \lambda_{2\ell} > 0$ (left), and $\lambda_{2\ell-1}\lambda_{2\ell} < 0$ (right).

Ξ_ℓ or $\Xi_{\ell-1} \rightarrow \Xi_\ell$ has two forms (Fig. 4.2), based upon Theorem 2.4 and Theorem 2.5 respectfully. The case $\lambda_{2\ell-1} > 0$ and $\lambda_{2\ell} > 0$ is easy to handle since the restrictions on tangent directions depend only on data points, more precisely on

$$\phi_{1,\ell-1} := \phi_1(\lambda_{2\ell-1}, \lambda_{2\ell}), \quad \phi_{2,\ell} := \phi_2(\lambda_{2\ell-1}, \lambda_{2\ell}), \quad \ell = 1, 2, \ldots, m.$$

The case $\lambda_{2\ell-1}\lambda_{2\ell} < 0$ is more complex since the existence conditions connect left and right tangent direction. For this reason, we introduce two additional maps, $\mathcal{R}_{1,\ell}(\mathcal{I})$, $\mathcal{R}_{2,\ell}(\mathcal{I})$, where $\mathcal{I}$ is an open or closed interval with endpoints a and b. For $\lambda_{2\ell-1} > 0$ and $\lambda_{2\ell} < 0$ the definition reads

$$\mathcal{R}_{1,\ell}(I) := \mathcal{R}_{1,\ell}(I; \lambda_{2\ell-1}, \lambda_{2\ell}) := \begin{cases} \emptyset; & b \le 0 \ \vee \ I = \emptyset, \\ (\phi_3(\lambda_{2\ell-1}, \lambda_{2\ell}, (a)_+), \phi_4(\lambda_{2\ell-1}, \lambda_{2\ell}, b)); & b > 0, \end{cases}$$

$$\mathcal{R}_{2,\ell}(I) := \mathcal{R}_{2,\ell}(I; \lambda_{2\ell-1}, \lambda_{2\ell}) := \begin{cases} \emptyset; & b \le 0 \ \vee \ a \ge \frac{1+2\lambda_{2\ell-1}}{\lambda_{2\ell-1}^2} \ \vee \ I = \emptyset, \\ (\phi_3(\lambda_{2\ell}, \lambda_{2\ell-1}, (a)_+), \phi_4(\lambda_{2\ell}, \lambda_{2\ell-1}, b)); & b < \frac{1}{\lambda_{2\ell-1}}, \\ (\phi_3(\lambda_{2\ell}, \lambda_{2\ell-1}, (a)_+), \infty); & b \ge \frac{1}{\lambda_{2\ell-1}}, \end{cases}$$

and for $\lambda_{2\ell-1} < 0$, $\lambda_{2\ell} > 0$ is given as

$$\mathcal{R}_{1,\ell}(I) := \mathcal{R}_{2,\ell}(I; \lambda_{2\ell}, \lambda_{2\ell-1}),$$
$$\mathcal{R}_{2,\ell}(I) := \mathcal{R}_{1,\ell}(I; \lambda_{2\ell}, \lambda_{2\ell-1}).$$

Recall Theorem 2.5 and Lemma 2.3. The meaning of $\mathcal{R}_{1,\ell}$ and $\mathcal{R}_{2,\ell}$ is the following. Suppose that $\mu_{2\ell-1}$, $\mu_{2\ell}$ are confined to intervals, i.e., $\mu_{2\ell-1} \in (a_1, b_1)$ and $\mu_{2\ell} \in (a_2, b_2)$. Then for every $\mu_{2\ell-1} \in (a_1, b_1) \cap \mathcal{R}_{2,\ell}((a_2, b_2))$ there exists at least one admissible $\mu_{2\ell} \in (a_2, b_2)$. Equivalently, for every $\mu_{2\ell} \in (a_2, b_2) \cap \mathcal{R}_{1,\ell}((a_1, b_1))$ there is at least one admissible $\mu_{2\ell-1} \in (a_1, b_1)$ (Fig. 4.3).

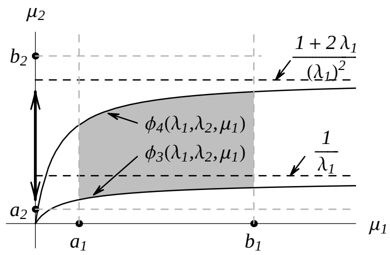

Figure 4.3: Geometric interpretation of $\mathcal{R}_{1,\ell}$ and $\mathcal{R}_{2,\ell}$ for $\ell = 1$. Every point (μ_1, μ_2) in the gray area is admissible.

Now, we can write the algorithm that should be called as

1. solution := $\emptyset$;
2. $\Xi := (\mathcal{I}_0)$;
3. ForwardSweep(m, Ξ, 1, solution);

procedure ForwardSweep(m, Ξ, ℓ, solution)
1. $S :=$ ForwardSplit(Ξ, ℓ);
2. **for** $i = 1$, $i \le \text{length}(S)$, $i = i + 1$

$\Xi_\ell := S_i$;
if $\ell = m$ **then** BackwardSweep(m, Ξ, ℓ, solution);
else ForwardSweep(m, Ξ, $\ell + 1$, solution);

procedure BackwardSweep(m, Ξ, ℓ, solution)
$\Xi_{\ell-1}$:= BackwardSplit(Ξ, ℓ);
if $\Xi_{\ell-1} \neq \emptyset$
if $\ell = 1$ **then** solution := solution $\cup \{\Xi\}$;
else BackwardSweep(m, Ξ, $\ell - 1$, solution);

procedure ForwardSplit(Ξ, ℓ)
$S := \emptyset$; $\mathcal{I} := \mu_{2\ell-1}(\Xi_{\ell-1})$; $\mathcal{J} := \emptyset$;
if $\lambda_{2\ell-1} > 0$ **and** $\lambda_{2\ell} > 0$ **then**
if $\mathcal{I} \leq \phi_{1,\ell-1}$ **then** $\mathcal{J} := \{(0, \phi_{2,\ell})\}$;
else if $\mathcal{I} \geq \phi_{1,\ell-1}$ **then** $\mathcal{J} := \{\,(\phi_{2,\ell}, \infty)\,\}$;
else $\mathcal{J} := \{\,(0, \phi_{2,\ell}), (\phi_{2,\ell}, \infty)\,\}$;
else if $\lambda_{2\ell-1} \cdot \lambda_{2\ell} < 0$ **then**
$\mathcal{J} := \{\mathcal{R}_{1,\ell}\,(\mathcal{I}; \lambda_{2\ell-1}, \lambda_{2\ell})\}$;
for $i = 1$, $i \leq$ length($\mathcal{J}$), $i = i + 1$
if $\mathcal{I} := \mu_{2\ell}^{-1}(\mathcal{J}_i) \cap \mathcal{I}_\ell \neq \emptyset$ **then** $S = S \cup \{\mathcal{I}\}$;
Return S

procedure BackwardSplit(Ξ, ℓ)
$\mathcal{I} := \mu_{2\ell}(\Xi_\ell)$; $\mathcal{J} := \emptyset$;
if $\lambda_{2\ell-1} > 0$ **and** $\lambda_{2\ell} > 0$ **then**
if $\mathcal{I} \leq \phi_{2,\ell}$ **then** $\mathcal{J} := (0, \phi_{1,\ell-1})$;
if $\mathcal{I} \geq \phi_{2,\ell}$ **then** $\mathcal{J} := (\phi_{1,\ell-1}, \infty)$;
else if $\lambda_{2\ell-1} \cdot \lambda_{2\ell} < 0$ **then**
$\mathcal{J} := \mathcal{R}_{2,\ell}\,(\mathcal{I}; \lambda_{2\ell-1}, \lambda_{2\ell})$;
Return $\mu_{2\ell-1}^{-1}(\mathcal{J}) \cap \Xi_{\ell-1}$;

The result of the algorithm *ForwardSweep* is a set called *solution*. It may be empty, if no admissible directions were found. If not, the elements of *solution* are vectors $\Xi = (\Xi_\ell)_{\ell=0}^m$, where each Ξ gives at least one admissible set of parameters $\xi_\ell \in \Xi_\ell$, $\ell = 0, 1, \ldots, m$. A brief look at Theorem 2.4 reveals that the result in the convex case is much stronger.

Corollary 4.2. *Suppose that the assumptions of Theorem 4.1 hold. Let*

$$\lambda_{2\ell-1} > 0,\ \lambda_{2\ell} > 0, \quad \ell = 1, 2, \ldots, m,$$

and let Ξ be a vector of intervals, returned by ForwardSweep. Any choice of parameters

$$(\xi_0, \xi_1, \ldots, \xi_m)\,, \quad \xi_\ell \in \Xi_\ell,$$

is admissible.

Even in the general case, there is a natural way to generate admissible choices, based upon the following consequence.

COROLLARY 4.3. *Suppose that the assumptions of Theorem 4.1 hold. Let* Ξ *be a vector of intervals, returned by ForwardSweep. For any* r*,* $0 \leq r \leq m$*, and any chosen* $\xi_r \in \Xi_r$*, one can find at least one admissible selection* $(\xi_0, \ldots, \xi_{r-1}, \xi_r, \xi_{r+1}, \ldots, \xi_m)$*,* $\xi_\ell \in \Xi_\ell$*.*

Let us now pick r, $1 \leq r \leq m-1$, and choose $\xi_r \in \Xi_r$. This means that Ξ_r has been in Ξ replaced by $[\xi_r, \xi_r]$. Corollary 4.3 for this new Ξ does not necessarily hold. But a call

$$\text{BackwardSweep}(r, \Xi, r, \text{solution})$$

shrinks properly the intervals $\Xi_{r-1}, \Xi_{r-2}, \ldots, \Xi_0$, and so does the mirror image of *BackwardSweep* on the intervals $\Xi_{r+1}, \Xi_{r+2}, \ldots, \Xi_m$. This brings the property of vector Ξ, described in Corollary 4.3, to each of its parts $(\Xi_\ell)_{\ell=0}^{r}$ and $(\Xi_\ell)_{\ell=r}^{m}$. So the whole step can be repeated on both parts separately. This *divide et impera* procedure can be repeated until we are left with an admissible solution. It adds at most a factor $\mathcal{O}(m)$ to the complexity of *ForwardSweep*.

Once the bounds Ξ have been determined, one has to choose the actual tangent directions. If $\boldsymbol{d}_{3\ell}$ are prescribed, Corollary 4.2 or the algorithm based upon Corollary 4.3 determines if the interpolation problem (1.2) and (1.3) has a solution. The same approach would work if the directions are approximated as

$$\boldsymbol{d}_{3\ell} = \boldsymbol{d}_{3\ell}\left(\gamma_\ell \underline{\xi}_\ell + (1-\gamma_\ell)\,\overline{\xi}_\ell\right), \quad \Xi_\ell = \left(\underline{\xi}_\ell, \overline{\xi}_\ell\right) \quad \text{or} \quad \Xi_\ell = \left[\underline{\xi}_\ell, \overline{\xi}_\ell\right],$$

where γ_ℓ may be determined by some local approximation scheme from the data (1.1) or simply chosen as a constant. Also, with the help of Ξ, one may look for a G^2 spline curve with $\boldsymbol{d}_{3\ell}$ determined implicitly as a solution of the system (1.4).

Let us conclude the paper with some numerical evidence. Fig 4.4 (a)–(b) shows a comparison between G^2 (dashed) and G^1 spline curve with tangent directions prescribed by $\gamma_\ell = \frac{1}{2}$ (light gray, dark gray), a choice that yields two solutions. Further, Fig 4.4 (c)–(d) shows G^2 (dashed) and G^1 spline curve, with directions determined by local quadratic interpolating polynomials based upon uniform (dark gray) and chord length parameterization (light gray). Note that the differences between the curves are very small and somewhere imperceptible. For the data in Fig 4.5 the G^2 spline curve can not be found, and interpolating polynomials based on uniform parameterization do not give admissible tangent directions, as can easily be checked. The difference between G^1 curve with $\gamma_\ell = \frac{1}{2}$ (dark gray) and quadratic chord length approximation (light gray) is again very small.

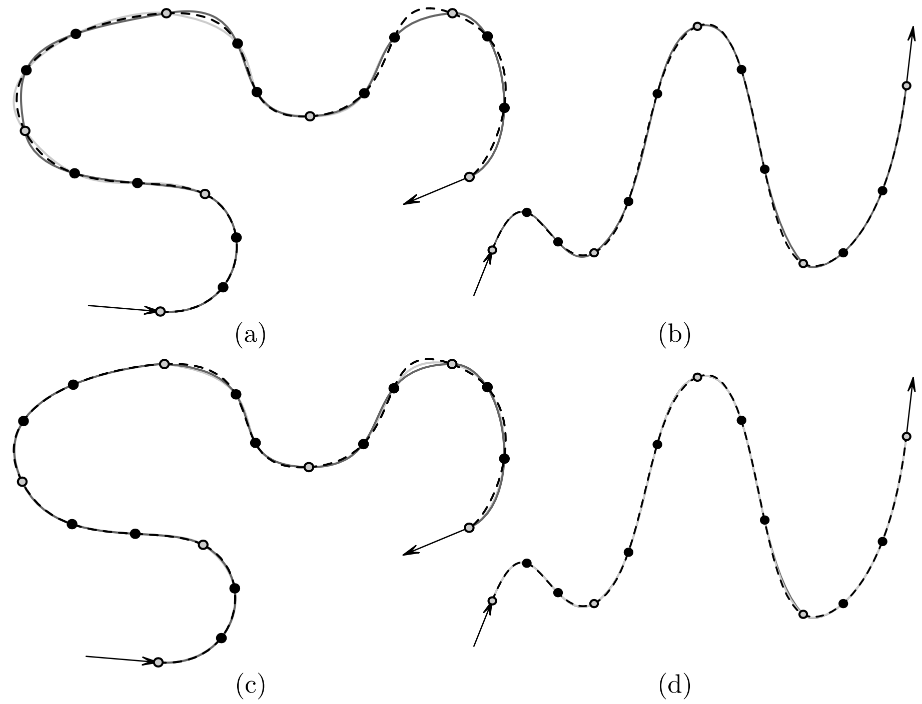

Figure 4.4: A comparison between G^1 (gray) and G^2 spline curves (dashed).

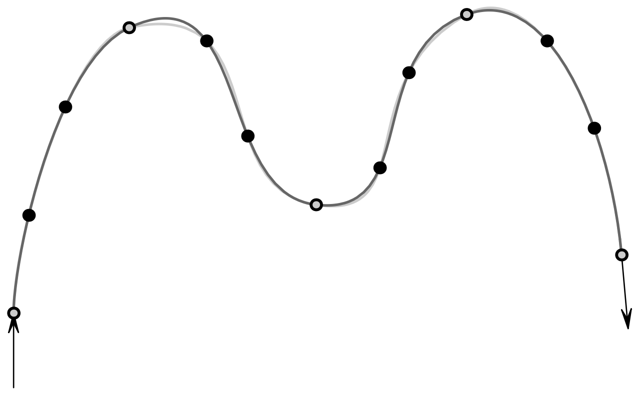

Figure 4.5: The comparison between G^1 spline curves at differently chosen tangent directions.

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