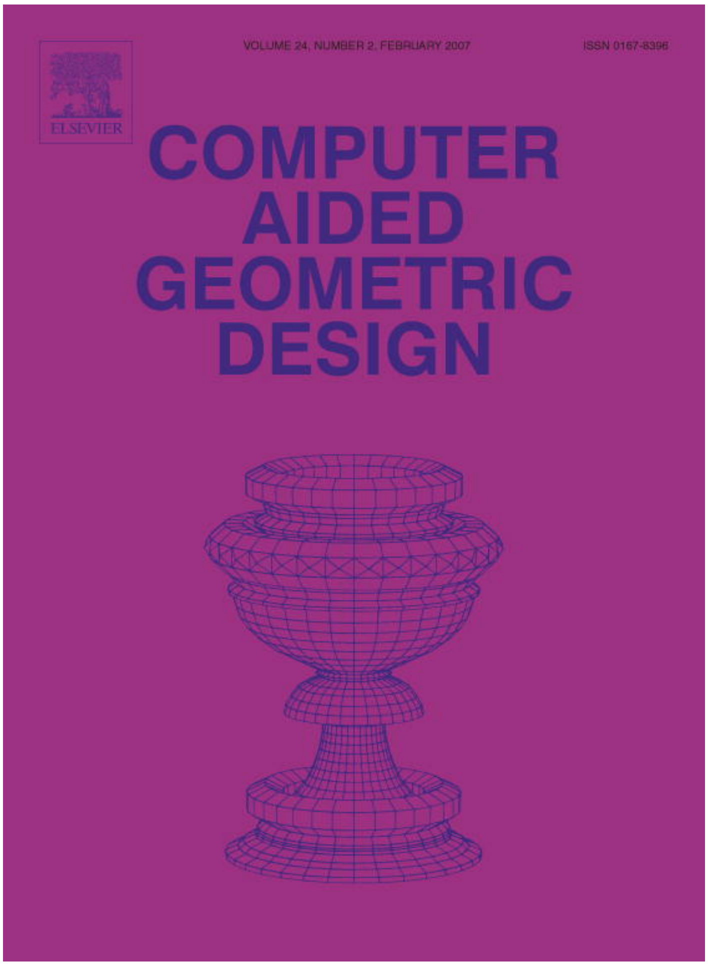
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Geometric interpolation by planar cubic polynomial curves

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Abstract

The purpose of this paper is to provide sufficient geometric conditions that imply the existence of a cubic parametric polynomial curve which interpolates six points in the plane. The conditions turn out to be quite simple and depend only on certain determinants derived from the data points.

Keywords: Polynomial curve; Geometric interpolation; Existence

1. Introduction

The geometric interpolation was introduced in (de Boor et al., 1987) as a Hermite cubic interpolation of two points, tangent directions and curvatures. It was shown that a planar convex curve can be approximated up to the sixth order accuracy. High approximation order is one of the reasons for the further work on the subject. The other is the fact that interpolating curve depends on geometric quantities (data points, tangent directions, curvatures, etc.) which are independent of parameterization. This places the geometric interpolation among important tools in the CAGD applications.

The geometric schemes include nonlinear equations, so the questions like existence of solution and efficient implementation have to be considered. That makes geometric schemes somewhat difficult to handle. The analysis is mainly done in an asymptotic way, i.e., the data are assumed to be sampled dense enough from a smooth curve (de Boor et al., 1987; Höllig and Koch, 1995, 1996; Jaklič et al., in press; Mørken and Scherer, 1997; Rababah, 1995; Scherer, 2000). Beside some special cases, like the interpolation of conic sections or of circular arcs (Jaklič et al., submitted for publication; Lyche and Mørken, 1994; Dokken et al., 1990; Floater, 1995, 1997; Mørken, 1991), there are only few results concerning geometric conditions that ensure the existence of the interpolant.

The interpolation by a parametric parabola at four distinct planar points was studied in (Lachance and Schwartz, 1991), where the conditions were established through geometric arguments. In (Mørken, 1995) the algebraic approach

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was applied, and results were extended to all possible cases (Taylor, Hermite, Lagrange). Perhaps the most general results can be found in (Kozak and Žagar, 2004), where the necessary and sufficient geometric conditions for the simplest nontrivial geometric interpolation schemes in all dimensions, i.e., the interpolation of $n+2$ distinct points in $\mathbb{R}^n$ by a polynomial curve of degree $\leqslant n$ are outlined.

In this paper, the Lagrange interpolation at six points in $\mathbb{R}^2$ by a cubic polynomial curve is studied and simple sufficient geometric conditions that ensure the existence of the interpolant are given. The problem is stated as follows. Let

$$\boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_5 \in \mathbb{R}^2, \quad \boldsymbol{T}_i \neq \boldsymbol{T}_{i+1},$$

be a given sequence of data points. Find a cubic parametric curve

$$\boldsymbol{P}_3 : [0,1] \to \mathbb{R}^2,$$

that interpolates these points at some values t_i ordered as

$$0 =: t_0 < t_1 < \cdots < t_5 := 1. \tag{1}$$

The admissible parameters t_i can be viewed as components of a point in the open simplex

$$\mathcal{D} := \left\{ \boldsymbol{t} := (t_i)_{i=0}^5;\ 0 =: t_0 < t_1 < \cdots < t_5 := 1 \right\},$$

with the boundary $\partial\mathcal{D}$ where at least two different t_i coincide. The nonlinear part of the problem is to determine the parameters $\boldsymbol{t} \in \mathcal{D}$ that satisfy

$$\boldsymbol{P}_3(t_i) = \boldsymbol{T}_i, \quad i = 0, 1, \ldots, 5. \tag{2}$$

Once the parameters t_i are determined, it is straightforward to obtain coefficients of $\boldsymbol{P}_3$ using any standard interpolation scheme componentwise.

As a motivation, let us compare the cubic geometric scheme with a componentwise quintic interpolation, where a parameterization is chosen in advance as the uniform, and the chord length parameterization. The cubic curve (black) clearly does the job much better than its quintic counterparts as one can observe in Fig. 1. The shape of the geometric interpolatory curve is as one would require for the given data points, without any visible extraneous inflections. Also, the computational effort to compute this six cubic interpolants turns out to be negligible. The Newton method with equidistant starting values $t_i = \frac{i}{5}$ converges within a machine precision accuracy in eight iterations on average.

There is perhaps a simple explanation to the fact that the cubic geometric interpolatory curves are superior. An approximate curvature, with denominator neglected, is a parabola

$$\det(\dot{\boldsymbol{P}}_3, \ddot{\boldsymbol{P}}_3),$$

so the rate of change of the curvature is approximately linear what pleases most the human eye.

2. The main results

The key role in this paper is played by the matrix of data differences,

$$\Delta\boldsymbol{T} := (\Delta\boldsymbol{T}_i)_{i=0}^4 \in \mathbb{R}^{2\times 5},$$

where $\Delta\boldsymbol{T}_i := \boldsymbol{T}_{i+1} - \boldsymbol{T}_i$, and by the signs and ratios of its minors

$$D_{i,j} := \det(\Delta\boldsymbol{T}_i, \Delta\boldsymbol{T}_j).$$

These are the volumes of parallelograms spanned by the vectors $\Delta\boldsymbol{T}_i$, $\Delta\boldsymbol{T}_j$. Let us define

$$\lambda_1 := \frac{D_{0,1}}{D_{1,2}}, \qquad \lambda_2 := \frac{D_{0,2}}{D_{1,2}}, \qquad \lambda_3 := \frac{D_{2,4}}{D_{2,3}}, \qquad \lambda_4 := \frac{D_{3,4}}{D_{2,3}}, \qquad \delta := \frac{D_{1,3}}{D_{1,2}}, \qquad \mu := \frac{D_{2,3}}{D_{1,2}},$$

$$\gamma_1 := \frac{\lambda_2(1+\lambda_2)}{\lambda_1(1+\lambda_2) + \sqrt{\lambda_1(1+\lambda_2)(\lambda_1+\lambda_2)}},$$

$$\gamma_2 := \frac{\lambda_3(1+\lambda_3)}{\lambda_4(1+\lambda_3) + \sqrt{\lambda_4(1+\lambda_3)(\lambda_3+\lambda_4)}}.$$

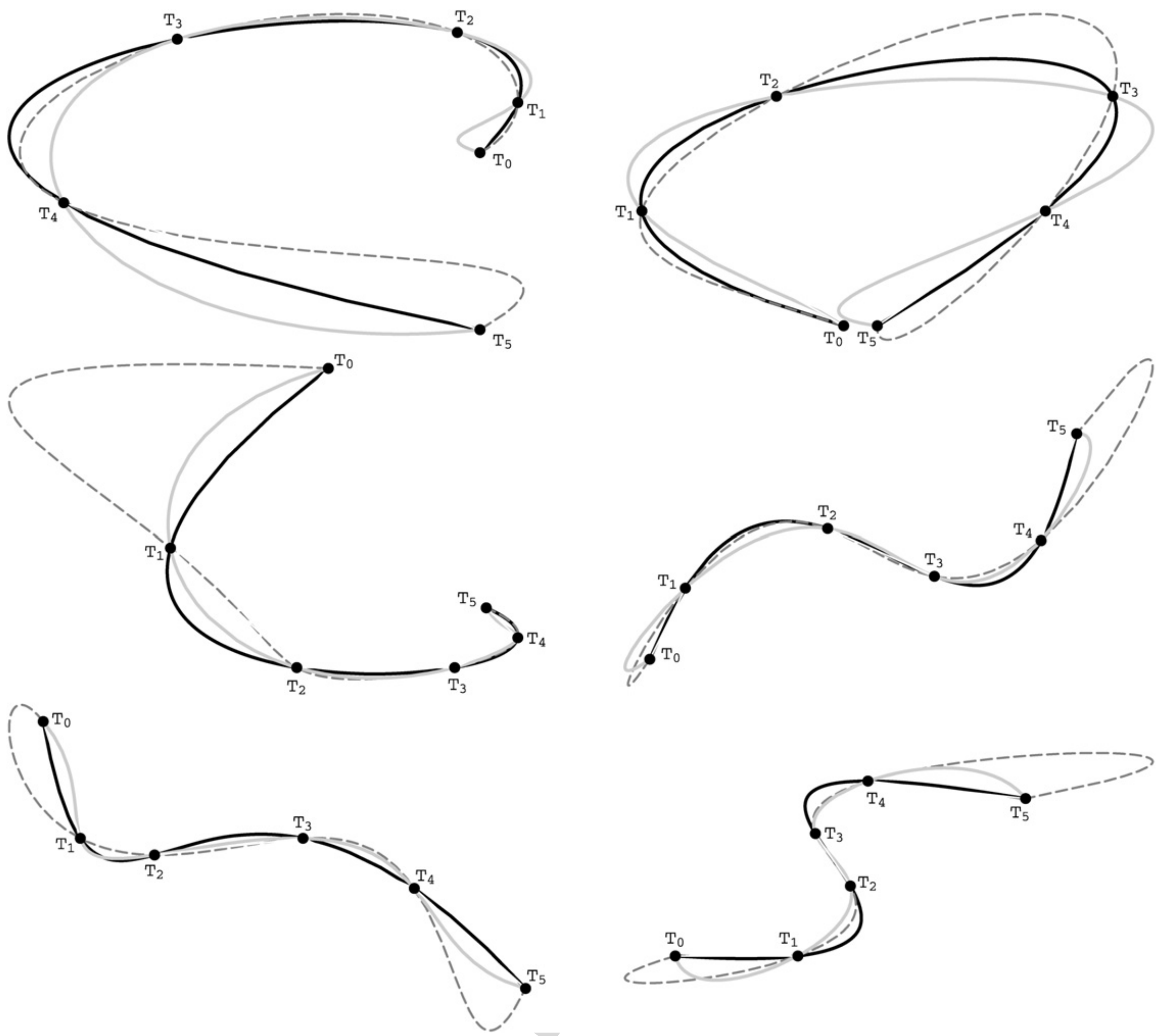

Fig. 1. A geometric cubic interpolant (black) and a quintic polynomial interpolating curves with uniform (grey) and chord length parameterization (dashed).

Note that the data points with a convex control polygon, as in the first three figures of Fig. 1, have $\mu > 0$ and $\lambda_i > 0$, $i = 1, 2, 3, 4$. The control polygons of the data points in the last three figures of Fig. 1 change from convexity to concavity at $\Delta \boldsymbol{T}_2$. Such data have $\lambda_i > 0$ and $\mu < 0$. We will restrict our study to these two types of data. Let us define $\boldsymbol{\lambda} := (\lambda_i)_{i=1}^4$, and the functions

$$\vartheta_1(\boldsymbol{\lambda}, \mu) := \frac{2\mu - \gamma_1 + \sqrt{\gamma_1^2 + 4\mu(1+\gamma_1)}}{2\gamma_1},$$

$$\vartheta_2(\boldsymbol{\lambda}, \mu) := \frac{2 - \mu\gamma_2 + \sqrt{\mu^2\gamma_2^2 + 4\mu(1+\gamma_2)}}{2\gamma_2},$$

$$\vartheta_3(\boldsymbol{\lambda}, \mu) := \frac{\lambda_1\mu}{\lambda_2} + \frac{\lambda_4}{\lambda_3} + \frac{\mu}{\lambda_2}\sqrt{\frac{\lambda_1(\lambda_1+\lambda_2)}{1+\lambda_2}},$$

$$\vartheta_4(\boldsymbol{\lambda}, \mu) := \frac{\lambda_1\mu}{\lambda_2} + \frac{\lambda_4}{\lambda_3} + \frac{1}{\lambda_3}\sqrt{\frac{\lambda_4(\lambda_3+\lambda_4)}{1+\lambda_3}},$$

that will be used in boundary relations between the constants, that ensure the existence of the solution. The main results of the paper are the following.

Theorem 1. *Suppose that $D_{1,2}D_{2,3} \neq 0$ and the data are convex, i.e., $\mu > 0$ and $\lambda_i > 0$, $i = 1, 2, 3, 4$. If either ϑ_ℓ are equal, $\vartheta_1(\boldsymbol{\lambda}, \mu) = \vartheta_2(\boldsymbol{\lambda}, \mu)$, or one of the following conditions is met,*

$$\delta < \min_{\ell=1,2}\{\vartheta_\ell(\boldsymbol{\lambda}, \mu)\} \quad \text{or} \quad \delta > \max_{\ell=1,2}\{\vartheta_\ell(\boldsymbol{\lambda}, \mu)\},$$

in case they are not equal then the interpolating curve $\boldsymbol{P}_3$ that satisfies (2) *exists.*

Theorem 2. *Suppose that $D_{1,2}D_{2,3} \neq 0$, and the data imply an inflection point, i.e., $\mu < 0$ and $\lambda_i > 0$ for all i. If*

$$\delta \in \big(\vartheta_3(\boldsymbol{\lambda}, \mu), \vartheta_4(\boldsymbol{\lambda}, \mu)\big),$$

then the interpolating curve $\boldsymbol{P}_3$ that satisfies (2) *exists.*

Theorems 1 and 2 provide only sufficient conditions for the existence of a cubic geometric interpolant. But the next conclusion excludes most of the data that do not satisfy this two theorems.

Theorem 3. *The cases where the solution of the interpolation problem* (2) *does not exist are summarized in Table* 1.

Some possibilities are not covered by Theorems 1, 2 or 3. As an example, consider the points

$$\boldsymbol{T}_0 = \begin{pmatrix} -20-\zeta \\ 3 \end{pmatrix}, \qquad \boldsymbol{T}_1 = \begin{pmatrix} -10 \\ 1 \end{pmatrix}, \qquad \boldsymbol{T}_2 = \begin{pmatrix} -5 \\ 0 \end{pmatrix},$$
$$\boldsymbol{T}_3 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \qquad \boldsymbol{T}_4 = \begin{pmatrix} 10 \\ 1 \end{pmatrix}, \qquad \boldsymbol{T}_5 = \begin{pmatrix} 20+\zeta \\ 3 \end{pmatrix}, \quad \zeta > 0, \tag{3}$$

with $\lambda_1 = \lambda_4 = -\frac{\zeta}{10}$, $\lambda_2 = \lambda_3 = 2$, $\delta = \mu = 1$. Note that neither the requirements of Theorems 1, 2 nor of Theorem 3 are met. Now, the data (3) admit two solutions for $\zeta \in (0, \zeta_0]$, where $\zeta_0 := 2.95373852$ (Fig. 2). For $\zeta = \zeta_0$ both of the solutions coincide with a cusp, but for $\zeta > \zeta_0$ no solution can be found.

Table 1

$D_{1,2}D_{2,3} \neq 0$		$D_{1,2}D_{2,3} = 0$
$\mu > 0$	$\mu < 0$	
$\lambda_2 \leqslant 0, \lambda_3 \leqslant 0$	$\lambda_2 \leqslant 0$	$D_{1,2} = 0,\ D_{2,3} = 0$
$\delta \leqslant 0, \lambda_1 \leqslant 0$	$\lambda_3 \leqslant 0$	$D_{1,2} = 0, \lambda_3 \leqslant 0$
$\delta \leqslant 0, \lambda_4 \leqslant 0$	$\lambda_1 \leqslant 0, \delta \leqslant 0$	$D_{2,3} = 0, \lambda_2 \leqslant 0$
$\lambda_1 \leqslant 0, \lambda_3 \leqslant 0, \lambda_4 \geqslant 0$	$\lambda_4 \leqslant 0, \delta \geqslant 0$	$D_{1,2} = 0, D_{0,1}D_{2,3} \geqslant 0$
$\lambda_2 \leqslant 0, \lambda_4 \leqslant 0, \lambda_1 \geqslant 0$		$D_{2,3} = 0, D_{1,2}D_{3,4} \geqslant 0$

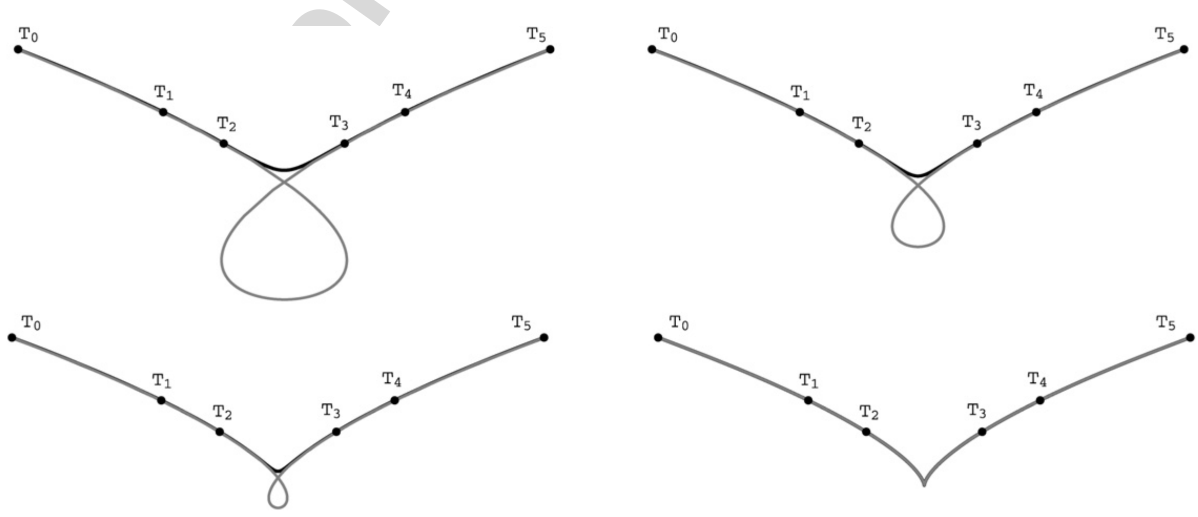

Fig. 2. Four cubic geometric interpolants at the points (3), with $\zeta = 2, 2.5, 2.8, \zeta_0$.

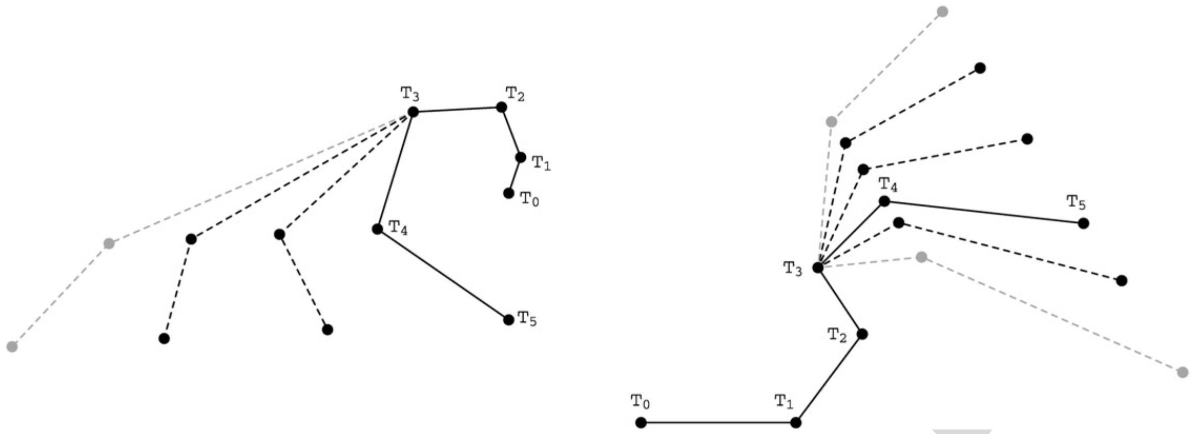

Fig. 3. The change of point positions as δ approaches $\vartheta_1(\boldsymbol{\lambda}, \mu)$ (left), and as δ changes from $\vartheta_3(\boldsymbol{\lambda}, \mu)$ to $\vartheta_4(\boldsymbol{\lambda}, \mu)$ (right).

The examples in Fig. 1 all satisfy the conditions of Theorem 1 or of Theorem 2. Let us look at two of them more precisely. In the first one $\delta < \vartheta_1(\boldsymbol{\lambda}, \mu)$. Fig. 3 (left) shows how the positions of points change as δ approaches $\vartheta_1(\boldsymbol{\lambda}, \mu)$. For $\delta \in [\vartheta_1(\boldsymbol{\lambda}, \mu), \vartheta_2(\boldsymbol{\lambda}, \mu)]$ no proper solution exists. Similarly, Fig. 3 (right) shows the displacement of points as δ changes from $\vartheta_3(\boldsymbol{\lambda}, \mu)$ to $\vartheta_4(\boldsymbol{\lambda}, \mu)$ for the last example of Fig. 1. For $\delta < \vartheta_3(\boldsymbol{\lambda}, \mu)$ or $\delta > \vartheta_4(\boldsymbol{\lambda}, \mu)$ two solutions $\boldsymbol{t} \in \mathcal{D}$ were found and the problem similar as in the example above has happened.

The requirements of Theorems 1 and 2 are quite simple, but the proof will take several steps. In Section 3 the system (2) will be transformed to a form more suitable for further analysis. In Section 4 it will be proved that any solution of (2) satisfying (1) cannot have the parameters t_i arbitrary close to the boundary $\partial\mathcal{D}$. A proof that the obtained nonlinear system has an odd number of solutions for particular data will be given in Section 5. Section 6 will extend this fact to the general case by a convex homotopy and the Brouwer's degree argument.

3. The equations

The divided difference $[t_\ell, t_{\ell+1}, \ldots, t_{\ell+4}]$, applied to the system (2), maps any $\boldsymbol{P}_3$ to zero. Let

$$\omega_\ell(t) := (t - t_\ell)(t - t_{\ell+1}) \ldots (t - t_{\ell+4}), \quad \dot{\omega}_\ell(t) = \frac{d}{dt}\omega_\ell(t), \quad \ell = 0, 1.$$

Since t_i are assumed to be distinct, one can express the divided difference in terms of $\dot{\omega}_\ell(t_i)$. The nonlinear part of the system (2), that should determine the unknowns t_1, t_2, t_3, t_4 thus becomes

$$\sum_{i=\ell}^{\ell+4} \frac{1}{\dot{\omega}_\ell(t_i)} \boldsymbol{T}_i = \boldsymbol{0}, \quad \ell = 0, 1. \tag{4}$$

Eqs. (4) were derived as necessary conditions for the existence of the solution of the interpolation problem (2), but they are sufficient too. A quintic polynomial curve $\boldsymbol{P}_5$ that solves the interpolation problem

$$\boldsymbol{P}_5(t_i) = \boldsymbol{T}_i, \quad i = 0, 1, \ldots, 5, \tag{5}$$

at distinct t_i is determined uniquely. But if $\boldsymbol{t} \in \mathcal{D}$ satisfies (4), one may apply $[t_\ell, t_{\ell+1}, \ldots, t_{\ell+4}]$, $\ell = 0, 1$, to both sides of (5). The right-hand side vanishes, so should the left one. This reveals that the quintic polynomial curve $\boldsymbol{P}_5$ in this case is actually a cubic one, the unique solution of (2). But

$$[t_\ell, t_{\ell+1}, \ldots, t_{\ell+4}]1 = \sum_{i=\ell}^{\ell+4} \frac{1}{\dot{\omega}_\ell(t_i)} = 0, \quad \ell = 0, 1, \tag{6}$$

and the system (4) can be rewritten as

$$(\boldsymbol{T}_i - \boldsymbol{T}_0)_{i=1}^4 \left(\frac{1}{\dot{\omega}_0(t_j)}\right)_{j=1}^4 = \boldsymbol{0}, \qquad (\boldsymbol{T}_5 - \boldsymbol{T}_{5-i})_{i=1}^4 \left(\frac{1}{\dot{\omega}_1(t_{5-j})}\right)_{j=1}^4 = \boldsymbol{0},$$

or, after inserting

$$\begin{pmatrix}1 & -1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix} = \mathrm{Id}$$

between the two factors, and using (6), as

$$(\Delta \boldsymbol{T}_i)_{i=\ell}^{\ell+3}\boldsymbol{\sigma}_\ell = \boldsymbol{0}, \quad \boldsymbol{\sigma}_\ell := \left(\sum_{i=\ell}^{\ell+j}\frac{1}{\dot\omega_\ell(t_i)}\right)_{j=0}^{3}, \quad \ell = 0, 1.$$

From now on let us assume that $D_{\ell+1,\ell+2} \neq 0$, $\ell = 0, 1$, as required in Theorems 1 and 2. The kernel of the matrix $(\Delta \boldsymbol{T}_i)_{i=\ell}^{\ell+3}$ is therefore two-dimensional, spanned by

$$\left(1, -\frac{D_{\ell,\ell+2}}{D_{\ell+1,\ell+2}}, \frac{D_{\ell,\ell+1}}{D_{\ell+1,\ell+2}}, 0\right)^T, \qquad \left(0, -\frac{D_{\ell+2,\ell+3}}{D_{\ell+1,\ell+2}}, \frac{D_{\ell+1,\ell+3}}{D_{\ell+1,\ell+2}}, -1\right)^T.$$

Since $\boldsymbol{\sigma}_\ell$ must be in the kernel,

$$\boldsymbol{\sigma}_\ell = a_\ell\begin{pmatrix}1\\ -\frac{D_{\ell,\ell+2}}{D_{\ell+1,\ell+2}}\\ \frac{D_{\ell,\ell+1}}{D_{\ell+1,\ell+2}}\\ 0\end{pmatrix} + b_\ell\begin{pmatrix}0\\ -\frac{D_{\ell+2,\ell+3}}{D_{\ell+1,\ell+2}}\\ \frac{D_{\ell+1,\ell+3}}{D_{\ell+1,\ell+2}}\\ -1\end{pmatrix}, \quad \ell = 0, 1, \tag{7}$$

for some a_ℓ and b_ℓ. After elimination of a_ℓ and b_ℓ,

$$a_\ell = \frac{1}{\dot\omega_\ell(t_\ell)}, \quad b_\ell = \frac{1}{\dot\omega_\ell(t_{\ell+4})}, \quad \ell = 0, 1,$$

and the use of (6), Eqs. (7) become

$$\frac{1}{\dot\omega_0(t_0)}(1+\lambda_2) + \frac{1}{\dot\omega_0(t_1)} + \frac{1}{\dot\omega_0(t_4)}\mu = 0, \tag{8}$$

$$\frac{1}{\dot\omega_0(t_0)}\lambda_1 + \frac{1}{\dot\omega_0(t_3)} + \frac{1}{\dot\omega_0(t_4)}(1+\delta) = 0, \tag{9}$$

$$\frac{1}{\dot\omega_1(t_1)}\left(1+\frac{\delta}{\mu}\right) + \frac{1}{\dot\omega_1(t_2)} + \frac{1}{\dot\omega_1(t_5)}\lambda_4 = 0, \tag{10}$$

$$\frac{1}{\dot\omega_1(t_1)}\frac{1}{\mu} + \frac{1}{\dot\omega_1(t_4)} + \frac{1}{\dot\omega_1(t_5)}(1+\lambda_3) = 0. \tag{11}$$

The system (8)–(11) is clearly equivalent to (4) since only nonsingular linear transformations were applied.

It will now be shown, that under certain restrictions the solutions $\boldsymbol{t} \in \mathcal{D}$ must stay aside from the boundary $\partial\mathcal{D}$. The theorem is stated as follows.

Theorem 4. *Suppose that the requirements of Theorems* 1 *or of Theorem* 2 *are met. Then the system* (8)–(11) *cannot have a solution arbitrary close to the boundary* $\partial\mathcal{D}$.

The proof of Theorem 4 is quite technical, and will be given as the next section.

4. Proof of Theorem 4

In order to prove Theorem 4 one has to show that

$$\Delta t_i := t_{i+1} - t_i \geqslant \text{const} > 0, \quad i = 0, 1, \dots, 4.$$

Here and throughout the rest of the paper, the term 'const' will stand for an arbitrary positive constant. Suppose that at least two parameters approach, i.e., $\Delta t_i \to 0$ for some i. There is enough to consider the following four possibilities:

Case 1: $\Delta t_0 \geqslant \text{const} > 0$, $\Delta t_4 \geqslant \text{const} > 0$,
Case 2: $\Delta t_0 \geqslant \text{const} > 0$, $\Delta t_4 \to 0$,
Case 3: $\Delta t_0 \to 0$, $\Delta t_4 \geqslant \text{const} > 0$,
Case 4: $\Delta t_0 \to 0$, $\Delta t_4 \to 0$.

In order to proceed, the following lemmas are needed.

Lemma 5. *Suppose that $\Delta t_i \to 0$, $i = 0, 3$, and $\Delta t_2 \geqslant \text{const} > 0$. Then*

$$\frac{\dot\omega_0(t_0)}{\dot\omega_0(t_4)} \to -\frac{\lambda_1}{\delta}.$$

Similarly, $\Delta t_i \to 0$, $i = 1, 4$, and $\Delta t_2 \geqslant const > 0$ imply

$$\frac{\dot\omega_1(t_5)}{\dot\omega_1(t_1)} \to -\frac{\mu\lambda_4}{\delta}.$$

Proof. Consider the first assertion. From

$$\frac{1}{t_3 - t_i} = \frac{1}{t_4 - t_i}\left(1 + \frac{\Delta t_3}{\Delta t_2 + (t_2 - t_i)}\right),$$

one obtains

$$\frac{1}{\dot\omega_0(t_3)} = -\frac{1}{\dot\omega_0(t_4)}\left(1 + \Delta t_3 \sum_{i=0}^{2} \frac{1}{\Delta t_2 + (t_2 - t_i)} + \mathcal{O}\big(\Delta t_3^2\big)\right).$$

Thus the expression

$$\frac{1}{\dot\omega_0(t_3)} + \frac{1}{\dot\omega_0(t_4)} = \prod_{i=0}^{2} \frac{1}{\Delta t_2 + \Delta t_3 + (t_2 - t_i)}\left(-\sum_{i=0}^{2} \frac{1}{\Delta t_2 + (t_2 - t_i)} + \mathcal{O}(\Delta t_3)\right)$$

stays bounded. Since $\dot\omega_0(t_0) \to 0$, Eq. (9) gives

$$\frac{\dot\omega_0(t_0)}{\dot\omega_0(t_4)} = -\frac{1}{\delta}\left(\lambda_1 + \dot\omega_0(t_0)\left(\frac{1}{\dot\omega_0(t_3)} + \frac{1}{\dot\omega_0(t_4)}\right)\right) \to -\frac{\lambda_1}{\delta}.$$

The second assertion follows similarly. □

Lemma 6. *Suppose that $\mu > 0$. Then $\Delta t_i \to 0$, $i = 0, 1, 2, 3$, implies $\delta > 0$ and $\delta \to \vartheta_1(\boldsymbol{\lambda}, \mu)$. Similarly, from $\Delta t_i \to 0$, $i = 1, 2, 3, 4$, it follows $\delta > 0$ and $\delta \to \vartheta_2(\boldsymbol{\lambda}, \mu)$.*

Proof. Let us prove the first assertion only. The proof of the second one is similar. After rewriting Eqs. (8)–(11) in a polynomial form, the last two equations simplify to

$$\delta \Delta t_2(\Delta t_2 + \Delta t_3) - \mu \Delta t_1(\Delta t_1 + 2\Delta t_2 + \Delta t_3) + \text{h.o.t.} = 0,$$
$$-\mu \Delta t_1(\Delta t_1 + \Delta t_2) + \Delta t_3(\Delta t_2 + \Delta t_3) + \text{h.o.t.} = 0,$$

where 'h.o.t.' stands for higher order terms that are small compared to the terms left in the expressions. Since $\Delta t_i > 0$, it is clear that $\delta > 0$. Moreover, by solving the first part of these two equations on Δt_2, Δt_3, the only admissible relation is

$$\Delta t_2 = \frac{\mu}{\delta}\left(1 + \sqrt{\frac{\delta + \mu}{\mu(1+\delta)}}\right)\Delta t_1 =: c_2 \Delta t_1, \qquad \Delta t_3 = \mu\sqrt{\frac{\delta + \mu}{\mu(1+\delta)}}\,\Delta t_1.$$

After substituting this into the remaining equations, we obtain

$$\Delta t_0^2 + (2 + c_2)\Delta t_0 \Delta t_1 - \lambda_2(1 + c_2)\Delta t_1^2 + \text{h.o.t.} = 0,$$
$$-\Delta t_0(\Delta t_0 + \Delta t_1) + \lambda_1 c_2(1 + c_2)\Delta t_1^2 + \text{h.o.t.} = 0.$$

Then, by the Gröbner basis one obtains an equivalent system

$$\begin{aligned}
&(1+c_2)\Delta t_1^2\big(c_2^2\lambda_1(1-\lambda_1)+2c_2\lambda_1(1+\lambda_2)-\lambda_2(1+\lambda_2)\big)+\text{h.o.t.}=0,\\
&-(1+c_2)\Delta t_1\big(\Delta t_0+(c_2\lambda_1-\lambda_2)\Delta t_1\big)+\text{h.o.t.}=0,\\
&-\Delta t_0(\Delta t_0+\Delta t_1)+c_2(1+c_2)\lambda_1\Delta t_1^2+\text{h.o.t.}=0.
\end{aligned}$$

Only particular constants will admit the solution of this system for small positive Δt_i. Since $c_2>0$, a straightforward computation shows that the solution exists only if $c_2\to\gamma_1$. Since $\lambda_1,\lambda_2>0$, it is easy to verify that $\gamma_1>0$. Therefrom by solving $c_2=\gamma_1$ on δ, one obtains $\delta\to\vartheta_1(\boldsymbol{\lambda},\mu)$, where $\vartheta_1(\boldsymbol{\lambda},\mu)>0$ as can easily be checked. □

Remark 7. Note that if $\vartheta_1(\boldsymbol{\lambda},\mu)=\vartheta_2(\boldsymbol{\lambda},\mu)$, the parameters t_i, $i=1,2,3,4$, cannot approach t_0 and t_5 at the same time.

Lemma 8. *Suppose that $\mu<0$. Then $\Delta t_i\to 0$, $i=0,1,2,4$, implies $\delta\to\vartheta_3(\boldsymbol{\lambda},\mu)$, and similarly, $\Delta t_i\to 0$, $i=0,2,3,4$, implies $\delta\to\vartheta_4(\boldsymbol{\lambda},\mu)$.*

Proof. Let us prove the second statement. After rewriting the equations (8)–(11) in a polynomial form, the last two equations simplify to

$$\begin{aligned}
&\lambda_4\Delta t_2(\Delta t_2+\Delta t_3)-\Delta t_4(\Delta t_3+\Delta t_4)+\text{h.o.t.}=0,\\
&\lambda_3\Delta t_3(\Delta t_2+\Delta t_3)-\Delta t_4(\Delta t_2+2\Delta t_3+\Delta t_4)+\text{h.o.t.}=0.
\end{aligned}$$

By solving the main part of these two equations on Δt_2, Δt_3, the only admissible relation is given as

$$\begin{aligned}
&\Delta t_2=\sqrt{\frac{1+\lambda_3}{\lambda_4(\lambda_3+\lambda_4)}}\,\Delta t_4=:c_2\Delta t_4,\\
&\Delta t_3=\frac{1}{\lambda_3}\left(1+\sqrt{\frac{\lambda_4(1+\lambda_3)}{\lambda_3+\lambda_4}}\right)\Delta t_4=:c_3\Delta t_4.
\end{aligned}$$

After substituting these expressions into the remaining equations, we obtain

$$\begin{aligned}
&\mu\Delta t_0+c_3(c_2+c_3)\lambda_2\Delta t_4^2+\text{h.o.t.}=0,\\
&(c_2\delta-c_3)\Delta t_0+c_2c_3(c_2+c_3)\lambda_1\Delta t_4^2+\text{h.o.t.}=0.
\end{aligned}$$

Again, with the help of the Gröbner basis, the equivalent system reads as

$$\begin{aligned}
&c_3(c_2+c_3)\Delta t_4^2\big(c_2(\lambda_2\delta-\lambda_1\mu)-c_3\lambda_2\big)+\text{h.o.t.}=0,\\
&\mu\Delta t_0+c_3(c_2+c_3)\lambda_2\Delta t_4^2+\text{h.o.t.}=0,\\
&(c_2\delta-c_3)\Delta t_0+c_2c_3(c_2+c_3)\lambda_1\Delta t_4^2+\text{h.o.t.}=0.
\end{aligned}$$

Since $c_2>0$, $c_3>0$, it is easy to verify that this system will have a solution for small Δt_i only if $\delta\to\vartheta_4(\boldsymbol{\lambda},\mu)$. The first statement is proved in a similar way. □

Now to the proof of Theorem 4. Note that

$$\operatorname{sign}\big(\dot\omega_\ell(t_i)\big)=(-1)^{\ell+i},\quad i=\ell,\ell+1,\ldots,\ell+4.$$

Also, from Eqs. (8)–(11) it is straightforward to derive a useful relation

$$\frac{\dot\omega_0(t_2)}{\dot\omega_0(t_3)}\frac{\dot\omega_1(t_3)}{\dot\omega_1(t_2)}=\frac{1+\delta+\lambda_1\frac{\dot\omega_0(t_4)}{\dot\omega_0(t_0)}}{\delta+\mu+(\lambda_1+\lambda_2)\frac{\dot\omega_0(t_4)}{\dot\omega_0(t_0)}}\,\frac{1+\frac{\delta}{\mu}+\lambda_4\frac{\dot\omega_1(t_1)}{\dot\omega_1(t_5)}}{\frac{1+\delta}{\mu}+(\lambda_3+\lambda_4)\frac{\dot\omega_1(t_1)}{\dot\omega_1(t_5)}}.\tag{12}$$

Case 1: In this case $\dot{\omega}_0(t_0) \geqslant \text{const} > 0$, $\dot{\omega}_1(t_5) \geqslant \text{const} > 0$. From Eqs. (8)–(11) it is straightforward to see that $\Delta t_1 \to 0$ or $\Delta t_2 \to 0$ implies $\Delta t_3 \to 0$. Consequently

$$\frac{\dot{\omega}_0(t_4)}{t_4 - t_1} = (1 - \Delta t_4)(\Delta t_2 + \Delta t_3)\Delta t_3 \to 0.$$

From (8) and (11) it is easy to derive

$$\frac{t_4 - t_1}{\dot{\omega}_0(t_4)} = \frac{1 + \lambda_2}{\mu} \frac{\Delta t_0 \Delta t_4}{\dot{\omega}_0(t_0)} + (1 + \lambda_3)\frac{(1 - \Delta t_0)\Delta t_4}{\dot{\omega}_1(t_5)}.$$

Since the right-hand side is bounded, but the left one is not, we have a contradiction that excludes the case 1.

Case 2: In this case $\dot{\omega}_0(t_0) \geqslant \text{const} > 0$, and $\dot{\omega}_1(t_5) \to 0$. Suppose first that $\Delta t_2 \geqslant \text{const} > 0$. Eq. (10) then implies $\Delta t_1 \to 0$. But then, (8) implies $\Delta t_3 \to 0$, and further

$$\dot{\omega}_0(t_4) = (1 - \Delta t_4)(\Delta t_1 + \Delta t_2 + \Delta t_3)(\Delta t_2 + \Delta t_3)\Delta t_3 \to 0.$$

Moreover, Eq. (9) yields

$$-\delta = 1 + \lambda_1 \frac{\dot{\omega}_0(t_4)}{\dot{\omega}_0(t_0)} + \frac{\dot{\omega}_0(t_4)}{\dot{\omega}_0(t_3)} = 1 + \frac{\dot{\omega}_0(t_4)}{\dot{\omega}_0(t_0)} - \prod_{i=0}^{2}\left(1 + \frac{\Delta t_3}{\Delta t_2 + (t_2 - t_i)}\right) \to 0.$$

Now, by Lemma 5 and the use of relation (12) one obtains

$$\frac{\dot{\omega}_0(t_4)}{\dot{\omega}_0(t_0)} \to 0, \qquad \frac{\dot{\omega}_1(t_1)}{\dot{\omega}_1(t_5)} \to 0, \qquad \frac{\dot{\omega}_0(t_2)}{\dot{\omega}_0(t_3)}\frac{\dot{\omega}_1(t_3)}{\dot{\omega}_1(t_2)} \to 1.$$

However, on the other hand,

$$\frac{\dot{\omega}_0(t_2)}{\dot{\omega}_0(t_3)}\frac{\dot{\omega}_1(t_3)}{\dot{\omega}_1(t_2)} = \frac{\Delta t_0 + \Delta t_1}{\Delta t_0 + \Delta t_1 + \Delta t_2}\frac{\Delta t_3 + \Delta t_4}{\Delta t_2 + \Delta t_3 + \Delta t_4} \to 0,$$

a contradiction. Therefore $\Delta t_2 \to 0$. But then Eqs. (8) and (9) imply $\Delta t_1 \to 0$, $\Delta t_3 \to 0$, $\mu > 0$, and Lemma 6 excludes the second case. The third case is a mirror view of the second one, and needs not to be proved.

Case 4: Here $\dot{\omega}_0(t_0), \dot{\omega}_1(t_5) \to 0$. Suppose again for a moment that $\Delta t_2 \geqslant \text{const} > 0$. Eqs. (9) and (10) then imply $\Delta t_1 \to 0$ and $\Delta t_3 \to 0$. So, by Lemma 5,

$$\frac{\dot{\omega}_0(t_0)}{\dot{\omega}_0(t_4)} \to -\frac{\lambda_1}{\delta}, \qquad \frac{\dot{\omega}_1(t_5)}{\dot{\omega}_1(t_1)} \to -\frac{\mu\lambda_4}{\delta}.$$

Therefrom by using the relation (12) we obtain

$$\frac{\dot{\omega}_0(t_2)\dot{\omega}_1(t_3)}{\dot{\omega}_1(t_2)\dot{\omega}_0(t_3)} \to \frac{\lambda_1}{\lambda_1\mu - \lambda_2\delta}\frac{\lambda_4\mu}{\lambda_4 - \lambda_3\delta} \neq 0,$$

but on the other hand

$$\frac{\dot{\omega}_0(t_2)}{\dot{\omega}_0(t_3)}\frac{\dot{\omega}_1(t_3)}{\dot{\omega}_1(t_2)} = \frac{\Delta t_0 + \Delta t_1}{\Delta t_0 + \Delta t_1 + \Delta t_2}\frac{\Delta t_3 + \Delta t_4}{\Delta t_2 + \Delta t_3 + \Delta t_4} \to 0.$$

Therefore $\Delta t_2 \to 0$. Suppose now that $\Delta t_1 \geqslant \text{const} > 0$. Eq. (8) gives

$$\dot{\omega}_0(t_4) = -\frac{\mu\dot{\omega}_0(t_0)}{1 + \lambda_2 + \frac{\dot{\omega}_0(t_0)}{\dot{\omega}_0(t_1)}} = -\frac{\mu\dot{\omega}_0(t_0)}{1 + \lambda_2 - \prod_{i=2}^{4}\left(1 + \frac{\Delta t_0}{t_i - t_2 + \Delta t_1}\right)} \to 0,$$

so $\Delta t_3 \to 0$, and $\mu < 0$. But by Lemma 8 this cannot happen. Similarly one can prove that $\Delta t_3 \geqslant \text{const} > 0$ implies $\Delta t_1 \to 0$, and $\mu < 0$. But, again by Lemma 8, this cannot happen either, which excludes the case 4, and therefore completes the proof of Theorem 4.

5. Particular case

Let us now consider the system (8)–(11) for particular data points

$$\boldsymbol{T}_0^* = \begin{pmatrix} 1-2c \\ 2 \end{pmatrix}, \qquad \boldsymbol{T}_1^* = \begin{pmatrix} -1-c \\ 1 \end{pmatrix}, \qquad \boldsymbol{T}_2^* = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$
$$\boldsymbol{T}_3^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{T}_4^* = \begin{pmatrix} 1+c \\ (-1)^s \end{pmatrix}, \qquad \boldsymbol{T}_5^* = \begin{pmatrix} -1+2c \\ 2(-1)^s \end{pmatrix}, \quad s = 0, 1, \tag{13}$$

where s determines whether μ^* is positive or negative, and $c = 0$ or $c = 6$ as will be needed in the proof of Theorem 1. It is straightforward to compute

$$\boldsymbol{\lambda}^* = \mathbf{1}, \qquad \delta^* = \frac{1}{2}\big(1 + (-1)^s\big)c, \qquad \mu^* = (-1)^s.$$

The number of all admissible solutions, i.e., solutions $\boldsymbol{t} \in \mathcal{D}$, is given in the next theorem.

Theorem 9. *Suppose the data points $\boldsymbol{T}_\ell$ are given by* (13) *with $c = 0$ or $c = 6$. The number of admissible solutions $\boldsymbol{t} \in \mathcal{D}$, counted with multiplicity, is odd. More precisely, the symmetric solution that satisfies $\Delta t_0 = \Delta t_4$, and $\Delta t_1 = \Delta t_3$, is unique. The number of the other solutions is even.*

Proof. The system (8)–(11) for the data points (13) simplifies to

$$\begin{aligned}
&\frac{2}{\dot\omega_0(t_0)} + \frac{1}{\dot\omega_0(t_1)} + (-1)^s \frac{1}{\dot\omega_0(t_4)} = 0,\\
&\frac{1}{\dot\omega_0(t_0)} + \frac{1}{\dot\omega_0(t_3)} + \left(1 + \frac{c}{2} + (-1)^s \frac{c}{2}\right)\frac{1}{\dot\omega_0(t_4)} = 0,\\
&\frac{2}{\dot\omega_1(t_5)} + \frac{1}{\dot\omega_1(t_4)} + (-1)^s \frac{1}{\dot\omega_1(t_1)} = 0,\\
&\frac{1}{\dot\omega_1(t_5)} + \frac{1}{\dot\omega_1(t_2)} + \left(1 + \frac{c}{2} + (-1)^s \frac{c}{2}\right)\frac{1}{\dot\omega_1(t_1)} = 0.
\end{aligned} \tag{14}$$

If there exists an admissible nonsymmetric solution $\boldsymbol{t} = (t_i)_{i=0}^5$, then $(1 - t_{5-i})_{i=0}^5$ is also an admissible solution, since

$$\dot\omega_0(1 - t_{5-i}) = \dot\omega_1(t_{5-i}), \quad i = 0, \ldots, 4, \qquad \dot\omega_1(1 - t_{5-i}) = \dot\omega_0(t_{5-i}), \quad i = 1, \ldots, 5.$$

Therefore the number of solutions, that are not symmetric, must be even. Let us examine the symmetric solutions now. It is easy to see that the first and the last two equations in (14) are then identical, and one is left with two equations

$$\begin{aligned}
&\frac{2t_3(t_3-1)(3t_4 - 2 + (-1)^s(1-t_4)) - 4t_4(2t_4-1)(t_4-1)}{t_3t_4(t_3-1)(t_4-1)(t_3-t_4)(t_3+t_4-1)(2t_4-1)} = 0,\\
&\frac{2t_4(t_3-t_4)(2t_3+2t_4-3) - c(t_3-1)(t_4-1)(2t_3-1)(1+(-1)^s)}{t_4(2t_3-1)(t_3-1)(t_4-1)(2t_4-1)(t_3-t_4)(t_3+t_4-1)} = 0,
\end{aligned}$$

for two unknowns ordered as $\frac{1}{2} < t_3 < t_4 < 1$. This yields a polynomial system that can be solved analytically. The admissible solution is unique (Table 2).

The proof of Theorem 9 is completed. □

Table 2
The admissible symmetric solution of the system (13)

	$s=0, c=0$	$s=0, c=6$	$s=1, c \in \mathbb{R}$
t_3	$\frac{1}{2}(3-\sqrt{3})$	$\frac{1}{44}(21 - 9\sqrt{3} + \sqrt{300 + 38\sqrt{3}})$	$\frac{3}{5}$
t_4	$\frac{\sqrt{3}}{2}$	$\frac{1}{44}(36 + 5\sqrt{3} - \sqrt{243 - 112\sqrt{3}})$	$\frac{9}{10}$

6. Proofs of Theorems 1, 2 and 3

In order to prove Theorems 1 and 2 one must show that the nonlinear system (8)–(11) has at least one solution $\boldsymbol{t} \in \mathcal{D}$. The convex homotopy will help us carry the conclusions from the particular to the general case.

Let us multiply (10) and (11) by μ and denote the obtained system (8)–(11) by $\boldsymbol{F}(\boldsymbol{t}; \boldsymbol{\lambda}, \delta, \mu) = \boldsymbol{0}$. Now, $\boldsymbol{F}$ can be split as $\boldsymbol{F}(\cdot; \boldsymbol{\lambda}, \delta, \mu) = \boldsymbol{F}_1(\cdot; \boldsymbol{\lambda}, \mu) + \delta \boldsymbol{F}_2$, where

$$\boldsymbol{F}_1(\cdot; \boldsymbol{\lambda}, \mu) := \boldsymbol{F}(\cdot; \boldsymbol{\lambda}, 0, \mu),$$
$$\boldsymbol{F}_2 := \boldsymbol{F}(\cdot; \boldsymbol{\lambda}, 1, \mu) - \boldsymbol{F}_1(\cdot; \boldsymbol{\lambda}, \mu).$$

The general data will be denoted by $(\boldsymbol{\lambda}, \delta, \mu)$, and the particular data (13), where s is chosen so that $\operatorname{sign}(\mu^*) = \operatorname{sign}(\mu)$, by $(\boldsymbol{\lambda}^*, \delta^*, \mu^*)$. The homotopy is now defined as

$$\boldsymbol{H}(\boldsymbol{t}; \alpha) := (1-\alpha)\boldsymbol{F}_1(\boldsymbol{t}; \boldsymbol{\lambda}^*, \mu^*) + \alpha \boldsymbol{F}_1(\boldsymbol{t}; \boldsymbol{\lambda}, \mu) + \varphi(\alpha, \delta^*, \delta)\boldsymbol{F}_2(\boldsymbol{t}),$$

where $\varphi(\cdot; \delta^*, \delta) : [0,1] \to \mathbb{R}$, satisfies $\varphi(0; \delta^*, \delta) = \delta^*$, $\varphi(1; \delta^*, \delta) = \delta$. Moreover, let

$$\boldsymbol{\lambda}(\alpha) := (1-\alpha)\boldsymbol{\lambda}^* + \alpha\boldsymbol{\lambda},$$
$$\delta(\alpha) := \varphi(\alpha, \delta^*, \delta),$$
$$\mu(\alpha) := (1-\alpha)\mu^* + \alpha\mu.$$

Then

$$\lambda_i(\alpha) \geqslant \min_{\alpha\in[0,1]} \big((1-\alpha)\lambda_i^* + \alpha\lambda_i\big) \geqslant \min\{\lambda_i^*, \lambda_i\} \geqslant \text{const} > 0,$$
$$\big|\mu(\alpha)\big| \geqslant \min_{\alpha\in[0,1]} \big|(1-\alpha)\mu^* + \alpha\mu\big| \geqslant \min\big\{|\mu^*|, |\mu|\big\} \geqslant \text{const} > 0.$$

Consider the case $\mu > 0$ as in Theorem 1 first. Note that $\vartheta_1(\boldsymbol{\lambda}^*, \mu^*) = \vartheta_2(\boldsymbol{\lambda}^*, \mu^*) = 4$. If $\delta < \min_{\ell=1,2}\{\vartheta_\ell(\boldsymbol{\lambda}, \mu)\}$ let us choose $c = \delta^* = 0$. It is then clear, that there exists a piecewise linear function $\varphi(\alpha, \delta^*, \delta)$, such that

$$\varphi(\alpha, \delta^*, \delta) < \min_{\ell=1,2}\big\{\vartheta_\ell\big(\boldsymbol{\lambda}(\alpha), \mu(\alpha)\big)\big\}, \quad \alpha \in [0,1].$$

Similarly we can do for $\delta > \max_{\ell=1,2}\{\vartheta_\ell(\boldsymbol{\lambda}, \mu)\}$ by choosing $c = \delta^* = 6$. In the case when $\mu < 0$, as in Theorem 2, we have

$$\vartheta_3(\boldsymbol{\lambda}^*, \mu^*) = -1 < \delta^* = 0 < \vartheta_4(\boldsymbol{\lambda}^*, \mu^*) = 1.$$

Since $\mu(\alpha) < 0$, it is straightforward to see that $\vartheta_3(\boldsymbol{\lambda}(\alpha), \mu(\alpha))$ and $\vartheta_4(\boldsymbol{\lambda}(\alpha), \mu(\alpha))$ cannot intersect for $\alpha \in [0,1]$. Thus there obviously exists a piecewise linear function $\varphi(\alpha, \delta^*, \delta)$, such that

$$\vartheta_3\big(\boldsymbol{\lambda}(\alpha), \mu(\alpha)\big) < \varphi(\alpha, \delta^*, \delta) < \vartheta_4\big(\boldsymbol{\lambda}(\alpha), \mu(\alpha)\big), \quad \alpha \in [0,1].$$

Therefore $\boldsymbol{H}(\boldsymbol{t}, \alpha) = \boldsymbol{0}$ meets the requirements of Theorem 4 for any $\alpha \in [0,1]$. As a consequence, a set of solutions

$$\mathcal{S} := \big\{\boldsymbol{t} \in \mathcal{D};\ \boldsymbol{H}(\boldsymbol{t}, \alpha) = \boldsymbol{0}\big\}$$

lies aside from the boundary $\partial\mathcal{D}$. More precisely, one can find a compact set $K \subset \mathcal{D}$, such that

$$\mathcal{S} \subset K \subset \mathcal{D}, \qquad \mathcal{S} \cap \partial K = \emptyset.$$

Thus the map $\boldsymbol{H}$ does not vanish at the boundary ∂K, and the Brouwer's degree (Berger, 1977) of $\boldsymbol{H}$ on K is invariant for all $\alpha \in [0,1]$. But by Theorem 9, it is odd for the particular map $\boldsymbol{F}(\cdot; \boldsymbol{\lambda}^*, \delta^*, \mu^*)$. Therefore $\boldsymbol{F}(\cdot; \boldsymbol{\lambda}, \delta, \mu) = \boldsymbol{0}$ must have at least one admissible solution and Theorems 1 and 2 are proved.

Let us now prove Theorem 3. Since the geometric interpolation is independent of the affine transformations of data points, one can choose the coordinate system so that one axis is in a direction of $\Delta\boldsymbol{T}_1$, $\Delta\boldsymbol{T}_2$ or $\Delta\boldsymbol{T}_3$. It is then straightforward to verify that the conditions of Theorem 3 imply that the other component of the interpolating curve as a cubic polynomial should have four zeros, what is a contradiction. The proof for the case $\mu > 0$, $\lambda_2 \leqslant 0$, and $\lambda_4 \leqslant 0$, is sketched in Fig. 4. The other cases follow by the same approach. This completes the proof of Theorem 3.

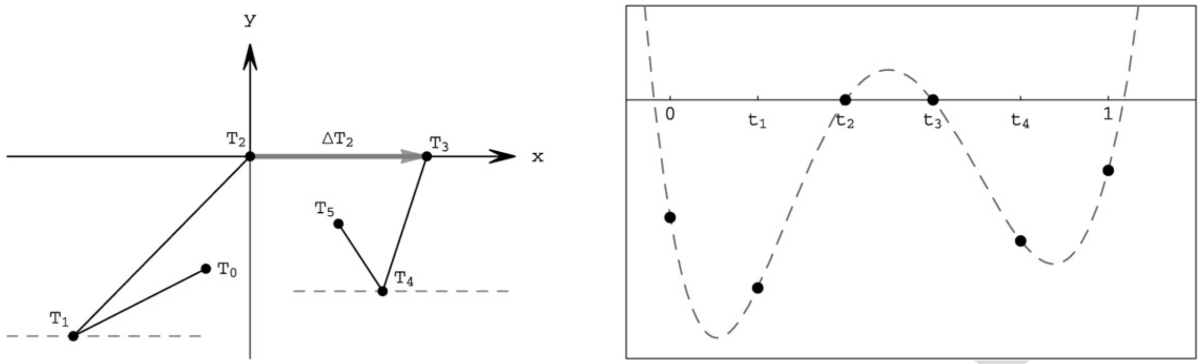

Fig. 4. The data points with $\mu > 0$, $\lambda_2 \leqslant 0$, $\lambda_4 \leqslant 0$ (left), and the y-component of $\boldsymbol{P}_3$ (right).

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