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Nonexistence of rational rotation-minimizing frames on cubic curves

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Abstract

We prove there is no rational rotation-minimizing frame (RMF) along any non-planar regular cubic polynomial curve. Although several schemes have been proposed to generate rational frames that approximate RMF's, exact rational RMF's have been only observed on certain Pythagorean-hodograph curves of degree seven. Using the Euler–Rodrigues frames naturally defined on Pythagorean-hodograph curves, we characterize the condition for the given curve to allow a rational RMF and rigorously prove its nonexistence in the case of cubic curves.

Keywords: Pythagorean-hodograph curve; Rotation-minimizing frame; Euler–Rodrigues frame; Rational frame; Cubic curve

1. Introduction

The rotation-minimizing frame (RMF), which executes the least possible rotation among all frames attached to a given curve in $\mathbb{R}^3$, finds important applications such as animation, swept surface constructions, and motion control. Classically, a pair of parameterized curves are said to be parallel if their corresponding tangent vectors are always parallel to each other. Together with the unit tangent vector, the normalized displacement vector connecting the corresponding points on the parallel curves constitutes the RMF of both the curves (Bishop, 1975). Explicitly speaking, if we denote the space curve by $\mathbf{r}(t)$, this normalized vector field is a solution to the differential equation (Klok, 1986)

$$\mathbf{f}'(t) = -\frac{\mathbf{f}(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^2}\mathbf{r}'(t).$$

Then, the triad of orthonormal vectors $\{\mathbf{u}, \mathbf{f}, \mathbf{u}\times\mathbf{f}\}$, where $\mathbf{u}(t)$ is the unit tangent of the curve $\mathbf{r}(t)$, forms an RMF of $\mathbf{r}(t)$.

The problem is that the above differential equation, which is in fact a system of linear differential equations, is quite difficult to solve exactly, and thus one usually resorts to approximate numerical methods (Jüttler and Wagner, 1999; Klok, 1986).

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One of the schemes proposed to approximate the RMF of a given curve is to approximate the curve itself by a series of planar segments that have naturally defined RMF's and then weave them into a continuous frame of the entire approximate curve (Wang and Joe, 1997).

Due to the preference of rational forms in computer graphics and computer-aided design applications, there have been several rational approximation schemes for RMF's (Farouki and Han, 2003; Jüttler and Mäurer, 1999; Mäurer and Jüttler, 1999). Since Pythagorean-hodograph (PH) curves—polynomial curves with polynomial speed (Farouki and Sakkalis, 1990)—are the only polynomial curves that allow rational frames, PH curves have been the principal platform for the rational approximation of the RMF.

The motion of the Frenet frame relative to an RMF is the rotation about the tangent vector with an angular velocity (when the curve is parameterized by arc length) equal to the torsion of the curve (Bishop, 1975). For PH curves, this angular deviation—the integral of the torsion—involves the integration of rational functions and thus generates not only rational terms but also transcendental terms (Farouki, 2003). Rational approximations to these transcendental terms offer an approach to the rational approximation of the RMF.

We can use other reference frames than the Frenet frame to construct the rational approximation of the RMF (Farouki and Han, 2003). The Euler–Rodrigues frame (ERF) arises from the quaternion representation of PH curves (Choi and Han, 2002), and one of its advantages over the Frenet frame is that it is a rational frame already. Furthermore, there exist PH curves of degree seven whose ERF is also an RMF. In fact the ERF on these PH curves is the only nontrivial (i.e., non-planar) example of rational RMF's ever reported. All previously mentioned schemes generate either a series of planar curves, an exact RMF which is not rational, or a rational frame which is not exactly rotation minimizing.

We will prove that the above options are the best that can be done for regular cubic polynomial curves. That is, we will show that there exists no rational rotation-minimizing frame on any regular cubic polynomial curve, unless the curve degenerates into a planar curve. Note that it was proved that no ERF, a special class of rational frames, on cubic PH curves can be rotation minimizing unless the curves are planar (Choi and Han, 2002). In this paper we will show that no rational frame at all can be rotation minimizing along any non-planar regular cubic PH curve.

We commence our discussion with a short review on quaternions using which the spatial PH curves and their ERF's are formulated.

2. Quaternion basics

Quaternions (Altmann, 1986) are a four-dimensional extension of complex numbers

$$\mathcal{A} = a + a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k},$$

whose multiplication is determined by the set of rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

It follows that the quaternion multiplication is noncommutative such that

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \qquad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

The product of $\mathcal{A}$ with its conjugate $\mathcal{A}^* = a - a_x\mathbf{i} - a_y\mathbf{j} - a_z\mathbf{k}$ always results in a nonnegative real number

$$\mathcal{A}\mathcal{A}^* = \mathcal{A}^*\mathcal{A} = a^2 + a_x^2 + a_y^2 + a_z^2.$$

Hence, every nonzero quaternion $\mathcal{A}$ has a multiplicative inverse $\mathcal{A}^{-1} = \mathcal{A}^*/(\mathcal{A}\mathcal{A}^*)$.

The three-dimensional space $\mathbb{R}^3$ is naturally embedded in quaternions by identifying each vector (a_x, a_y, a_z) as the (pure) quaternion $a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$. This identification proves to be quite useful when we consider rotations in $\mathbb{R}^3$. Any unit quaternion $\mathcal{U}$ (i.e., $\mathcal{U}\mathcal{U}^* = 1$) can be written in the form

$$\mathcal{U} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{n},$$

for some real θ and unit vector $\mathbf{n}$ (identified as a quaternion). Then, the mapping in $\mathbb{R}^3$ defined by $\mathbf{v} \mapsto \mathcal{U}\mathbf{v}\mathcal{U}^*$ is the rotation of $\mathbf{v}$ about the vector $\mathbf{n}$ by the angle θ. It follows that for a generic nonzero quaternion $\mathcal{A}$, the mapping $\mathbf{v} \mapsto \mathcal{A}\mathbf{v}\mathcal{A}^*$ is a scaled rotation in $\mathbb{R}^3$; a vector $\mathbf{v}$ is rotated by the unit quaternion $\mathcal{U} = \mathcal{A}/\sqrt{\mathcal{A}\mathcal{A}^*}$ and then multiplied by the scalar $\mathcal{A}\mathcal{A}^*$.

3. Rational frames on polynomial curves

Given a curve $\mathbf{r}(t)$ in $\mathbb{R}^3$, a frame on $\mathbf{r}(t)$ is a correspondence that assigns to each t an ordered ternary of orthonormal vectors $\{\mathbf{f}_1(t), \mathbf{f}_2(t), \mathbf{f}_3(t)\}$. We call a frame adapted if one of the vectors (usually designated to $\mathbf{f}_1$) is the unit tangent vector $\mathbf{r}'(t)/\|\mathbf{r}'(t)\|$. In this paper we only consider adapted frames and simply call them frames.

For a polynomial curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, the unit tangent vector is rational if and only if $\|\mathbf{r}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$ is a polynomial. It follows that the Pythagorean-hodograph (PH) condition (Farouki et al., 2002)

$$x'(t)^2 + y'(t)^2 + z'(t)^2 = \sigma(t)^2$$

for some polynomial $\sigma(t)$ is a necessary condition for the existence of rational frames on $\mathbf{r}(t)$.

For a curve $\mathbf{r}(t)$ to have a frame defined everywhere, the curve should be regular, i.e., $\mathbf{r}'(t) \neq \mathbf{0}$ at every t. A polynomial curve $\mathbf{r}(t)$ is regular if and only if $x'(t), y'(t), z'(t)$ do not share a common linear factor. The following theorem is a slight refinement of the previous characterization (Choi et al., 2002) of regular PH curves in terms of quaternion polynomials.

Theorem 1. *A polynomial curve $\mathbf{r}(t)$ is a regular PH curve if and only if*

$$\mathbf{r}'(t) = \mathcal{A}(t)\mathbf{i}\mathcal{A}(t)^* \tag{1}$$

for some quaternion polynomial $\mathcal{A}(t) = u(t) + v(t)\mathbf{i} + p(t)\mathbf{j} + q(t)\mathbf{k}$, where $u(t)$, $v(t)$, $p(t)$, $q(t)$ do not share a common linear factor.

Remark 2. The linear common factor condition is essential here. In general, $\gcd(u, v, p, q) = \text{constant}$ does not guarantee $\gcd(x', y', z') = \text{constant}$ (Farouki et al., 2004).

If $\mathcal{A}(t)$ is a degree-n quaternion polynomial, the PH curve defined by (1) is a degree-$(2n + 1)$ polynomial. Hence, if there exists a polynomial curve that allows a rational frame to be defined everywhere on it, the curve should be a regular PH curve of odd degree satisfying the condition of Theorem 1. In the subsequent discussion of rational frames, therefore, we only need to consider regular PH curves of odd degree.

Since the mapping $\mathbf{v} \mapsto \mathcal{A}(t)\mathbf{v}\mathcal{A}(t)^*$ is a scaled rotation in $\mathbb{R}^3$ for each t, the following vectors

$$\mathbf{g}_1(t) = \frac{\mathcal{A}(t)\mathbf{i}\mathcal{A}(t)^*}{\mathcal{A}(t)\mathcal{A}(t)^*}, \qquad \mathbf{g}_2(t) = \frac{\mathcal{A}(t)\mathbf{j}\mathcal{A}(t)^*}{\mathcal{A}(t)\mathcal{A}(t)^*}, \qquad \mathbf{g}_3(t) = \frac{\mathcal{A}(t)\mathbf{k}\mathcal{A}(t)^*}{\mathcal{A}(t)\mathcal{A}(t)^*}$$

form a right-handed rational frame along the PH curve $\mathbf{r}(t)$ defined by (1). This is the so-called Euler–Rodrigues frame (ERF) (Choi and Han, 2002) of $\mathbf{r}(t)$, and $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ are the column vectors of the following matrix

$$\frac{1}{u^2 + v^2 + p^2 + q^2} \begin{bmatrix} u^2 + v^2 - p^2 - q^2 & 2(vp - uq) & 2(up + vq) \\ 2(uq + vp) & u^2 - v^2 + p^2 - q^2 & 2(pq - uv) \\ 2(vq - up) & 2(uv + pq) & u^2 - v^2 - p^2 + q^2 \end{bmatrix}.$$

Given a PH curve $\mathbf{r}(t)$ its quaternion polynomial $\mathcal{A}(t)$ satisfying (1) is not unique (Farouki et al., 2002). If $\mathcal{B}(t)$ is another quaternion polynomial such that

$$\mathcal{B}(t)\mathbf{i}\mathcal{B}(t)^* = \mathcal{A}(t)\mathbf{i}\mathcal{A}(t)^*,$$

we have

$$\mathcal{A}^{-1}\mathcal{B}\mathbf{i}\mathcal{B}^*(\mathcal{A}^{-1})^* = \mathbf{i}. \tag{2}$$

If we denote $\mathcal{Q} = \mathcal{A}^{-1}\mathcal{B}$, Eq. (2) can be written in the form

$$\mathcal{Q}\mathbf{i}\mathcal{Q}^* = \mathbf{i},$$

whose general solution is

$$\mathcal{Q}(\theta) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{i}$$

for any real θ. It follows that for each regular PH curve $\mathbf{r}(t)$, we have a one-parameter family of quaternion polynomials $\mathcal{A}_\theta(t) := \mathcal{A}(t)\mathcal{Q}(\theta)$ generating the same $\mathbf{r}(t)$ by the formula $\mathbf{r}'(t) = \mathcal{A}_\theta(t)\mathbf{i}\mathcal{A}_\theta(t)^*$. This is an adaptation of the Hopf fibration (Lyons, 2003) to the PH curve formulation (Choi et al., 2002). Since the quaternion $\mathcal{Q}(\theta)$ rotates the vectors $\mathbf{j}$ and $\mathbf{k}$ by θ about the $\mathbf{i}$-axis, any pair of the ERF's, corresponding to $\mathcal{A}_{\theta_1}$ and $\mathcal{A}_{\theta_2}$, maintain a fixed angle $|\theta_1 - \theta_2|$ along the entire curve $\mathbf{r}(t)$.

Suppose there exists another right-handed rational frame $\{\mathbf{f}_1(t), \mathbf{f}_2(t), \mathbf{f}_3(t)\}$ on $\mathbf{r}(t)$ with $\mathbf{f}_1$ being the unit tangent of $\mathbf{r}(t)$. Then we have

$$\begin{aligned} \mathbf{g}_1(t) &= \mathbf{f}_1(t), \\ \mathbf{g}_2(t) &= \mathbf{f}_2(t)\cos\phi(t) + \mathbf{f}_3(t)\sin\phi(t), \\ \mathbf{g}_3(t) &= \mathbf{f}_3(t)\cos\phi(t) - \mathbf{f}_2(t)\sin\phi(t) \end{aligned} \tag{3}$$

for some $\phi(t)$. Since $\mathbf{f}_i(t)$ and $\mathbf{g}_j(t)$ are all rational, their coefficients $\cos\phi(t)$ and $\sin\phi(t)$ are also rational. Hence, we may write

$$\cos\phi(t) = \frac{\alpha(t)}{\gamma(t)}, \qquad \sin\phi(t) = \frac{\beta(t)}{\gamma(t)}$$

for some polynomials $\alpha(t), \beta(t), \gamma(t)$ with $\gcd(\alpha, \beta, \gamma) = \text{constant}$. Since they form a Pythagorean triple, i.e., $\alpha(t)^2 + \beta(t)^2 = \gamma(t)^2$, we can further assume that $\alpha(t)$ and $\beta(t)$ are relatively prime. Then, there exist relatively prime polynomials $a(t)$ and $b(t)$ satisfying (Farouki and Sakkalis, 1990)

$$\alpha(t) = a(t)^2 - b(t)^2, \qquad \beta(t) = 2a(t)b(t), \qquad \gamma(t) = a(t)^2 + b(t)^2.$$

We may conclude as follows.

Theorem 3. *Any rational frame $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ on a regular PH curve $\mathbf{r}(t)$ is of the form*

$$\begin{aligned} \mathbf{f}_1 &= \mathbf{g}_1, \\ \mathbf{f}_2 &= \frac{a^2 - b^2}{a^2 + b^2}\mathbf{g}_2 - \frac{2ab}{a^2 + b^2}\mathbf{g}_3, \\ \mathbf{f}_3 &= \frac{a^2 - b^2}{a^2 + b^2}\mathbf{g}_3 + \frac{2ab}{a^2 + b^2}\mathbf{g}_2 \end{aligned} \tag{4}$$

for some relatively prime polynomials $a(t)$ and $b(t)$, where $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ is the ERF of $\mathbf{r}(t)$.

Remark 4. If we define $\mathcal{Q}(t) = a(t) - b(t)\mathbf{i}$, the frame $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ on $\mathbf{r}(t)$ is the ERF of the curve $\tilde{\mathbf{r}}(t)$ defined by $\tilde{\mathbf{r}}'(t) = [\mathcal{A}(t)\mathcal{Q}(t)]\mathbf{i}[\mathcal{A}(t)\mathcal{Q}(t)]^*$. Since $\tilde{\mathbf{r}}'(t) = [a(t)^2 + b(t)^2]\mathbf{r}'(t)$, the curve $\tilde{\mathbf{r}}(t)$ is a regular PH curve of degree-$[2(m+n)+1]$, where m is the degree of $\mathcal{Q}(t)$. The curves $\mathbf{r}(t)$ and $\tilde{\mathbf{r}}(t)$ have the same unit tangent vector, or the tangent indicatrix.

4. Rotation-minimizing frames

While the vectors $\mathbf{f}_2(t)$ and $\mathbf{f}_3(t)$ of the frame are required to stay on the plane perpendicular to $\mathbf{r}'(t)$, they are free to rotate on that plane. Among all possible frames on $\mathbf{r}(t)$, there exists a frame whose $\mathbf{f}_2$ and $\mathbf{f}_3$ experience the least possible rotation. Such a frame is called the rotation-minimizing frame (RMF) and its $\mathbf{f}_2$ and $\mathbf{f}_3$ are the solutions of the linear differential equation (Klok, 1986)

$$\mathbf{f}'(t) = -\frac{\mathbf{f}(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^2}\mathbf{r}'(t).$$

Note that such a pair $\mathbf{f}_2$ and $\mathbf{f}_3$ is not unique; there exist a one-parameter family of RMF's corresponding to different sets of initial position of $\mathbf{f}_2$ and $\mathbf{f}_3$.

On the other hand, it is easier to verify whether a given frame is an RMF. A frame $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is an RMF if and only if either of (and thus each of) $\mathbf{f}_2'(t)$ and $\mathbf{f}_3'(t)$ is parallel to $\mathbf{f}_1(t)$, the unit tangent vector (Bishop, 1975). Equivalently,

$$\mathbf{f}_2'(t)\cdot\mathbf{f}_3(t) \equiv 0 \tag{5}$$

is the necessary-and-sufficient condition for the frame to be rotation minimizing.

Suppose the rational frame (4) is an RMF. By differentiating (4) and invoking the fact (Choi and Han, 2002)

$$\mathbf{g}_2' \cdot \mathbf{g}_3 = 2\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2},$$

it can be verified immediately that condition (5) is equivalent to

$$\frac{ab' - a'b}{a^2 + b^2} = \frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2}. \tag{6}$$

Theorem 5. *The regular PH curve $\mathbf{r}(t)$ defined by a quaternion polynomial $\mathcal{A}(t) = u(t) + v(t)\mathbf{i} + p(t)\mathbf{j} + q(t)\mathbf{k}$ has a rational RMF if and only if there exist relatively prime polynomials $a(t)$ and $b(t)$ satisfying* (6).

5. Cubic PH curves

We write the linear quaternion polynomial $\mathcal{A}(t)$ for a cubic PH curve $\mathbf{r}(t)$ as follows:

$$u(t) = u_0 + u_1 t, \qquad v(t) = v_0 + v_1 t, \qquad p(t) = p_0 + p_1 t, \qquad q(t) = q_0 + q_1 t,$$

or

$$\mathcal{A}(t) = \mathcal{A}_0 + \mathcal{A}_1 t, \quad \mathcal{A}_i = u_i + v_i\mathbf{i} + p_i\mathbf{j} + q_i\mathbf{k}, \quad i = 0, 1.$$

Then, we have

$$uv' - u'v - pq' + p'q = u_0v_1 - u_1v_0 - p_0q_1 + p_1q_0$$

and

$$u^2 + v^2 + p^2 + q^2 = \langle \mathcal{A}, \mathcal{A} \rangle = \|\mathcal{A}_0\|^2 + 2t\langle \mathcal{A}_0, \mathcal{A}_1 \rangle + t^2\|\mathcal{A}_1\|^2, \tag{7}$$

where we identified quaternions as four-dimensional vectors and defined their inner product by $\langle \mathcal{A}, \mathcal{B} \rangle = ab + a_x b_x + a_y b_y + a_z b_z$. (Note also that $\|\mathcal{A}\|^2 = \mathcal{A}\mathcal{A}^* = \langle \mathcal{A}, \mathcal{A} \rangle$.) The discriminant of the above quadratic polynomial is

$$\Delta = \sqrt{\|\mathcal{A}_0\|^2\|\mathcal{A}_1\|^2 - \langle \mathcal{A}_0, \mathcal{A}_1 \rangle^2} = \sqrt{A^2 + B^2 + C^2 + D^2 + E^2 + F^2}, \tag{8}$$

where

$$\begin{aligned} A &= u_0v_1 - u_1v_0, \qquad & C &= u_0p_1 - u_1p_0, \qquad & E &= u_0q_1 - u_1q_0, \\ B &= p_0q_1 - p_1q_0, \qquad & D &= v_0q_1 - v_1q_0, \qquad & F &= v_0p_1 - v_1p_0. \end{aligned} \tag{9}$$

Note that $\Delta > 0$ since $\mathbf{r}(t)$ is regular. Then Eq. (7) can be written as

$$u^2 + v^2 + p^2 + q^2 = \|\mathcal{A}_1\|^2\left[t + \frac{\langle \mathcal{A}_0, \mathcal{A}_1 \rangle + i\Delta}{\|\mathcal{A}_1\|^2}\right]\left[t + \frac{\langle \mathcal{A}_0, \mathcal{A}_1 \rangle - i\Delta}{\|\mathcal{A}_1\|^2}\right],$$

and we have

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} = \frac{u_0v_1 - u_1v_0 - p_0q_1 + p_1q_0}{\|\mathcal{A}_1\|^2\left[t + \frac{\langle \mathcal{A}_0, \mathcal{A}_1 \rangle + i\Delta}{\|\mathcal{A}_1\|^2}\right]\left[t + \frac{\langle \mathcal{A}_0, \mathcal{A}_1 \rangle - i\Delta}{\|\mathcal{A}_1\|^2}\right]}. \tag{10}$$

Now consider the prime factor decomposition of the complex polynomial

$$a(t) + ib(t) = \lambda(t - \mu_1)\ldots(t - \mu_m).$$

Since $a(t)$ and $b(t)$ are relatively prime real polynomials, we have $\mu_j \neq \bar{\mu}_k$ for any $1 \leqslant j, k \leqslant m$ (including the case $j = k$). Then

$$a'(t) - ib'(t) = \bar{\lambda}\sum_{j=1}^{m}\left[\prod_{\substack{k=1 \\ k \neq j}}^{m}(t - \bar{\mu}_k)\right].$$

It follows that

$$aa' + bb' - i(ab' - a'b) = (a + ib)(a' - ib') = |\lambda|^2 \sum_{j=1}^{m} \left[(t - \mu_j) \prod_{\substack{k=1 \\ k \neq j}}^{m} |t - \mu_k|^2 \right].$$

By taking the imaginary part, we obtain

$$ab' - a'b = |\lambda|^2 \sum_{j=1}^{m} \left[\operatorname{Im}(\mu_j) \prod_{\substack{k=1 \\ k \neq j}}^{m} |t - \mu_k|^2 \right].$$

If we divide both sides by $a(t)^2 + b(t)^2 = |\lambda|^2 |t - \mu_1|^2 \dots |t - \mu_m|^2$, we finally have

$$\frac{ab' - a'b}{a^2 + b^2} = \frac{\operatorname{Im}(\mu_1)}{|t - \mu_1|^2} + \cdots + \frac{\operatorname{Im}(\mu_m)}{|t - \mu_m|^2}. \tag{11}$$

Now, if we compare the rational functions of Eqs. (10) and (11), they can be the same expressions only if they have the same poles, where the rational functions diverge. Thus, we can readily see that

$$\mu_j = -\frac{\langle \mathcal{A}_0, \mathcal{A}_1 \rangle + i\Delta}{\|\mathcal{A}_1\|^2} \quad \text{or} \quad -\frac{\langle \mathcal{A}_0, \mathcal{A}_1 \rangle - i\Delta}{\|\mathcal{A}_1\|^2}, \quad j = 1, \dots, m \tag{12}$$

is the necessary condition for Eq. (6) to hold. Since no pair of μ_j's can be conjugate to each other, the choices in (12) should be uniform throughout all $j = 1, \dots, m$. Then, after the common factors are canceled out, condition (6) is reduced to

$$\pm m\Delta = u_0 v_1 - u_1 v_0 - p_0 q_1 + p_1 q_0. \tag{13}$$

We found an inequality related to (13).

Lemma 6. *With Δ defined by* (8)*, we have*

$$\Delta \geqslant |u_0 v_1 - u_1 v_0| + |p_0 q_1 - p_1 q_0|.$$

Proof. We have

$$\Delta^2 - \big(|u_0 v_1 - u_1 v_0| + |p_0 q_1 - p_1 q_0|\big)^2 = C^2 + D^2 + E^2 + F^2 - 2|AB|, \tag{14}$$

where A, B, C, D, E, F are as defined in (9). It is easy to verify that $AB = CD - EF$. Then

$$C^2 + D^2 + E^2 + F^2 - 2|AB| \geqslant 2\big(|CD| + |EF| - |CD - EF|\big) \geqslant 0. \qquad \square$$

If we combine Lemma 6 and Eq. (13), we obtain

$$\Delta \geqslant |u_0 v_1 - u_1 v_0| + |p_0 q_1 - p_1 q_0| \geqslant |(u_0 v_1 - u_1 v_0) - (p_0 q_1 - p_1 q_0)| = m\Delta. \tag{15}$$

Condition (15) cannot be satisfied unless $m = 0$ or 1. If $m = 0$, the last equality in (15) requires

$$u_0 v_1 - u_1 v_0 - p_0 q_1 + p_1 q_0 = 0,$$

the condition that forces the cubic PH curve to be planar (Choi and Han, 2002). A rational RMF can be trivially defined on any planar PH curve.

Now if $m = 1$, every inequality that should be satisfied in (15) is now an equality. That is, we must have

$$\Delta = |u_0 v_1 - u_1 v_0| + |p_0 q_1 - p_1 q_0| = \big|(u_0 v_1 - u_1 v_0) - (p_0 q_1 - p_1 q_0)\big|. \tag{16}$$

From the second equality in (16) we must have

$$(u_0 v_1 - u_1 v_0)(p_0 q_1 - p_1 q_0) \leqslant 0.$$

In view of (14) the first equality in (16) can now be written as

$$C^2 + D^2 + E^2 + F^2 + 2AB = 0. \tag{17}$$

On the other hand, the curvature of a generic PH curve can be simplified to produce (Farouki, 2003)

$$\kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{2\sqrt{\rho}}{\|\mathbf{r}'\|^2},$$

where

$$\rho = (up' - u'p)^2 + (vq' - v'q)^2 + (uq' - u'q)^2 + (vp' - v'p)^2 + 2(uv' - u'v)(pq' - p'q).$$

For cubic PH curves, the above expression has the form

$$\rho = C^2 + D^2 + E^2 + F^2 + 2AB.$$

Hence, in view of (17), when $m = 1$, condition (15) can be satisfied only if $\mathbf{r}(t)$ is a straight line. We may conclude our discussion as follows.

Theorem 7. *If a regular cubic PH curve allows a rational RMF, the curve should be planar.*

Remark 8. The case $m = 1$ can be also explained using the previous result on ERF (Choi and Han, 2002). If there exist a pair of linear polynomials $a(t)$ and $b(t)$ such that the frame $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ in (4) is rotation minimizing, the ERF of $\tilde{\mathbf{r}}(t)$ defined in Remark 4 is also rotation minimizing. But $\tilde{\mathbf{r}}(t)$ is a quintic PH curve, whose ERF can be an RMF if and only if it is planar.

6. Concluding remarks

Regarding the construction of moving frames along space curves, the rotation-minimizing property and the rational dependence upon the curve parameter are two of the most desirable characteristics among others. The only reported instances satisfying both have been the Euler–Rodrigues frames on certain Pythagorean-hodograph curves of degree seven. In this paper, a characterization of regular polynomial curves that allow rational rotation-minimizing frames has been developed, and the case of cubic curves has been thoroughly examined resulting in the nonexistence of such frames. This approach can be extended to polynomial curves of higher degrees to facilitate the understanding of the existence and characterization of their rational rotation-minimizing frames.

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