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Quintic space curves with rational rotation-minimizing frames

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ABSTRACT

The existence of polynomial space curves with rational rotation-minimizing frames (RRMF curves) is investigated, using the Hopf map representation for PH space curves in terms of complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$. The known result that all RRMF cubics are degenerate (linear or planar) curves is easily deduced in this representation. The existence of non-degenerate RRMF quintics is newly demonstrated through a constructive process, involving simple algebraic constraints on the coefficients of two quadratic complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ that are sufficient and necessary for any PH quintic to admit a rational rotation-minimizing frame. Based on these constraints, an algorithm to construct RRMF quintics is formulated, and illustrative computed examples are presented. For RRMF quintics, the Bernstein coefficients $\boldsymbol{\alpha}_0$, $\boldsymbol{\beta}_0$ and $\boldsymbol{\alpha}_2$, $\boldsymbol{\beta}_2$ of the quadratics $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ may be freely assigned, while $\boldsymbol{\alpha}_1$, $\boldsymbol{\beta}_1$ are fixed (modulo one scalar freedom) by the constraints. Thus, RRMF quintics have sufficient freedoms to permit design by the interpolation of G^1 Hermite data (end points and tangent directions). The methods can also be extended to higher-order RRMF curves.

1. Introduction

An *adapted frame* on a space curve $\mathbf{r}(t)$ is an orthonormal basis for $\mathbb{R}^3$ such that, at each curve point, the unit tangent $\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$ is one basis vector, and the other two basis vectors span the curve *normal plane*. The *Frenet frame* $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ defined by the curve intrinsic geometry is perhaps the most familiar example — the principal normal $\mathbf{n}$ points to the *center of curvature*, and the binormal is defined by $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ (Kreyszig, 1959). However, as noted by Bishop (1975), there is an infinitude of adapted frames associated with any given space curve, and among them the *rotation-minimizing frames* (RMFs) are useful in animation, motion planning, swept surface constructions, and related applications where the Frenet frame may prove unsuitable (Guggenheimer, 1989; Jüttler, 1998; Klok, 1986; Sír and Jüttler, 2005; Wang and Joe, 1997; Wang et al., 2008).

The variation of an adapted orthonormal frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ with $\mathbf{f}_1 = \mathbf{t}$ along a curve $\mathbf{r}(t)$ may be specified by its vector angular velocity $\boldsymbol{\omega}(t)$ as

$$\mathbf{f}_1' = \boldsymbol{\omega} \times \mathbf{f}_1, \qquad \mathbf{f}_2' = \boldsymbol{\omega} \times \mathbf{f}_2, \qquad \mathbf{f}_3' = \boldsymbol{\omega} \times \mathbf{f}_3.$$

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The magnitude and direction of $\boldsymbol{\omega}$ specify the instantaneous angular speed $\omega = |\boldsymbol{\omega}|$ and rotation axis $\mathbf{a} = \boldsymbol{\omega}/|\boldsymbol{\omega}|$ of the frame vectors $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$. Since $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ constitute an orthonormal basis for $\mathbb{R}^3$ we can write

$$\boldsymbol{\omega} = \omega_1 \mathbf{f}_1 + \omega_2 \mathbf{f}_2 + \omega_3 \mathbf{f}_3,$$

where the components of $\boldsymbol{\omega}$ are given by

$$\omega_1 = \mathbf{f}_3 \cdot \mathbf{f}_2' = -\mathbf{f}_2 \cdot \mathbf{f}_3', \qquad \omega_2 = \mathbf{f}_1 \cdot \mathbf{f}_3' = -\mathbf{f}_3 \cdot \mathbf{f}_1', \qquad \omega_3 = \mathbf{f}_2 \cdot \mathbf{f}_1' = -\mathbf{f}_1 \cdot \mathbf{f}_2'. \tag{1}$$

The distinguishing feature of a rotation-minimizing adapted frame is that $\boldsymbol{\omega}$ maintains a zero component ω_1 along $\mathbf{f}_1 = \mathbf{t}$, i.e., $\boldsymbol{\omega} \cdot \mathbf{t} \equiv 0$. This means that, at every point of $\mathbf{r}(t)$, there is no instantaneous rotation of the normal-plane vectors $\mathbf{f}_2$ and $\mathbf{f}_3$ about $\mathbf{f}_1$. The focus of this paper is on curves for which the RMF vectors have a *rational* dependence on the curve parameter.

There is an intimate connection between the *Pythagorean-hodograph* (PH) *curves* — i.e., polynomial curves $\mathbf{r}(t) = (x(t), y(t), z(t))$ that satisfy

$$\left|\mathbf{r}'(t)\right| = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)} = \sigma(t) \tag{2}$$

for some polynomial $\sigma(t)$ — and curves with rational RMFs. Namely, since satisfaction of (2) is necessary for a rational unit tangent, the search for curves with rational RMFs may be restricted to PH curves.[1] For a comprehensive review of the theory and applications of PH curves, see Farouki (2008).

Rational forms are always preferred in computer-aided design whenever possible, since they are exactly compatible with the representation schemes of most CAD systems and permit efficient computations. In general, however, both Frenet frames and rotation-minimizing frames are not rational — even for PH curves. Choi and Han (2002) observed that the spatial PH curves always admit a rational adapted frame, the so-called *Euler–Rodrigues frame* (ERF). Although the ERF does not have an intuitive geometric significance, and is dependent upon the chosen Cartesian coordinates, it has the advantage over the Frenet frame of being non-singular at inflection points.

These facts have motivated recent interest in two special classes of PH curves — the *double Pythagorean-hodograph* (DPH) *curves* (Beltran and Monterde, 2007; Farouki et al., 2009a, 2009b) which have rational Frenet frames (and thus might also be called RFF curves), and the set of PH curves with *rational RMFs* (Choi and Han, 2002; Han, 2008). For brevity, we shall call the latter RRMF curves — bearing in mind that they are necessarily PH curves. DPH curves are intimately related to the theory of *helical* polynomial curves (Beltran and Monterde, 2007; Farouki et al., 2004; Monterde, 2009): it was shown in Beltran and Monterde (2007) that all helical polynomial curves must be DPH curves, although there exist non-helical DPH curves of degree 7 or more.

Investigations of RRMF curves have thus far been relatively sparse. Choi and Han (2002) studied conditions under which the ERF of a PH curve coincides with an RMF, and showed that, for PH cubics, the ERF and Frenet frame are equivalent; for PH quintics, the ERF can be rotation-minimizing only in the degenerate case of planar curves; and the simplest non-planar PH curves for which the ERF can be an RMF are of degree 7. More recently, Han (2008) presented an algebraic criterion characterizing RRMFs of any (odd) degree, and showed that RRMF cubics are degenerate — i.e., they are either planar PH curves, or PH curves with non-primitive hodographs.

In this paper, the existence of non-degenerate quintic RRMF curves is demonstrated through a simple constructive procedure, based on a detailed analysis of the algebraic condition for rationality of the RMF on a PH curve in the Hopf map representation. This analysis furnishes a simple complex-arithmetic algorithm for the practical construction of RRMF quintics, and permits generalization to the study of higher-order RRMF curves.

The plan for the paper is as follows. After some preliminaries concerning PH curve representations and adapted frames in Section 2, the condition for existence of rational RMFs is formulated and analyzed in terms of the Hopf map form of spatial PH curves in Section 3. This condition is then analyzed in detail, in the context of PH cubics and quintics, in Sections 4 and 5. The Hopf map representation offers a simple proof of the fact that only linear or planar cubics admit rational RMFs. For quintics, simple constraints that characterize the existence of rational RMFs are derived, leading to an easily-implemented algorithm. Finally, Section 6 briefly discusses the generalization of these results to higher-order RRMF curves, while Section 7 summarizes and assesses the main results of this paper.

2. Adapted frames on spatial PH curves

A polynomial space curve $\mathbf{r}(t)$ is a *Pythagorean-hodograph* (PH) *curve* if its derivative $\mathbf{r}'(t)$ satisfies (2) for some polynomial $\sigma(t)$. Two alternative (but equivalent) algebraic characterizations for hodographs $\mathbf{r}'(t)$ that satisfy this condition were introduced by Choi et al. (2002). In the *quaternion representation*, a spatial Pythagorean hodograph is generated from a quaternion polynomial $\mathcal{A}(t) = u(t) + v(t)\mathbf{i} + p(t)\mathbf{j} + q(t)\mathbf{k}$ by the expression

$$\mathbf{r}'(t) = \mathcal{A}(t)\mathbf{i}\mathcal{A}^*(t), \tag{3}$$

[1] It is possible to compute exact RMFs on spatial PH curves (Farouki, 2002), although in general they incur transcendental functions. As an alternative, piecewise-rational RMF approximations for PH curves have been proposed in Farouki and Han (2003).

$\mathcal{A}^*(t)$ being the quaternion conjugate of $\mathcal{A}(t)$ — note that the expression on the right is a quaternion with zero real (scalar) part, which may be regarded as a vector in $\mathbb{R}^3$. The *Hopf map representation*, on the other hand, generates Pythagorean hodographs from pairs of complex polynomials[2] $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ by the expression

$$\mathbf{r}'(t) = \left(\left|\boldsymbol{\alpha}(t)\right|^2 - \left|\boldsymbol{\beta}(t)\right|^2, 2\,\mathrm{Re}\big(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t)\big), 2\,\mathrm{Im}\big(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t)\big)\right). \tag{4}$$

The equivalence of (3) and (4) may be seen by taking $\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\boldsymbol{\beta}(t)$, where the imaginary unit i is identified with the quaternion element $\mathbf{i}$. For a comprehensive treatment of these two representations, see Choi et al. (2002), Farouki (2008).

Curves with primitive hodographs $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ — satisfying $\gcd(x'(t), y'(t), z'(t)) = \text{constant}$ — are typically preferred in practice, since a common real root of the hodograph components can incur a cusp on the curve. In the Hopf map form (4) of PH curves, the hodograph is primitive if and only if $\gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) = \text{constant}$ (see Remarks 1 and 2 in Farouki et al. (2009a)).

This paper relies more on the Hopf map form (4), since it proves better suited to the problem at hand, although the quaternion form (3) is also used occasionally. The PH curve defined by integrating (4) is evidently of (odd) degree, $n = 2m + 1$, where $m = \max(\deg(\boldsymbol{\alpha}(t)), \deg(\boldsymbol{\beta}(t)))$. The polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ are represented here in Bernstein form:

$$\boldsymbol{\alpha}(t) = \sum_{j=0}^{m} \boldsymbol{\alpha}_j \binom{m}{j} (1-t)^{m-j} t^j, \qquad \boldsymbol{\beta}(t) = \sum_{j=0}^{m} \boldsymbol{\beta}_j \binom{m}{j} (1-t)^{m-j} t^j. \tag{5}$$

Consider an adapted frame $(\mathbf{f}_1(t), \mathbf{f}_2(t), \mathbf{f}_3(t))$ on a regular curve $\mathbf{r}(t)$, with

$$\mathbf{f}_1(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Many adapted frames exist, since a rotation of the normal-plane vectors by an angle $\phi(t)$ defines a new adapted frame upon replacing $\mathbf{f}_2(t)$, $\mathbf{f}_3(t)$ by

$$\cos\phi(t)\,\mathbf{f}_2(t) - \sin\phi(t)\,\mathbf{f}_3(t), \qquad \sin\phi(t)\,\mathbf{f}_2(t) + \cos\phi(t)\,\mathbf{f}_3(t).$$

The frame $(\mathbf{f}_1(t), \mathbf{f}_2(t), \mathbf{f}_3(t))$ is rotation-minimizing if and only if its angular velocity $\boldsymbol{\omega}(t)$ maintains a zero component ω_1, given by (1), along $\mathbf{f}_1(t)$ (Bishop, 1975).

Now if we desire *rational* adapted frames, we may consider only PH curves — since condition (2) is necessary for $\mathbf{f}_1(t)$ to be rational. A rational adapted frame $(\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t))$ known (Choi and Han, 2002) as the Euler–Rodrigues frame (ERF) can be defined on any spatial PH curve in terms of the quaternion representation. This frame is obtained by unitizing $\mathcal{A}(t)\mathbf{i}\mathcal{A}^*(t)$, $\mathcal{A}(t)\mathbf{j}\mathcal{A}^*(t)$, $\mathcal{A}(t)\mathbf{k}\mathcal{A}^*(t)$ — in terms of the Hopf map form, it is given by

$$\begin{aligned}
\mathbf{e}_1 &= \frac{((|\boldsymbol{\alpha}|^2 - |\boldsymbol{\beta}|^2, 2\,\mathrm{Re}(\boldsymbol{\alpha}\overline{\boldsymbol{\beta}}), 2\,\mathrm{Im}(\boldsymbol{\alpha}\overline{\boldsymbol{\beta}}))}{|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2}, \\
\mathbf{e}_2 &= \frac{(-2\,\mathrm{Re}(\boldsymbol{\alpha}\boldsymbol{\beta}), \mathrm{Re}(\boldsymbol{\alpha}^2 - \boldsymbol{\beta}^2), \mathrm{Im}(\boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2))}{|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2}, \\
\mathbf{e}_3 &= \frac{(2\,\mathrm{Im}(\boldsymbol{\alpha}\boldsymbol{\beta}), -\mathrm{Im}(\boldsymbol{\alpha}^2 - \boldsymbol{\beta}^2), \mathrm{Re}(\boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2))}{|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2}.
\end{aligned} \tag{6}$$

Hence, any other adapted frame on a spatial PH curve, defined by a rotation $\phi(t)$ of $\mathbf{e}_2(t)$, $\mathbf{e}_3(t)$ in the curve normal plane, is rational if and only if

$$\cos\phi(t) = \frac{P_1(t)}{P_3(t)}, \qquad \sin\phi(t) = \frac{P_2(t)}{P_3(t)},$$

for real polynomials $P_1(t)$, $P_2(t)$, $P_3(t)$ satisfying

$$\gcd(P_1(t), P_2(t), P_3(t)) = \text{constant} \quad \text{and} \quad P_1^2(t) + P_2^2(t) = P_3^2(t).$$

Hence, relatively prime polynomials $a(t)$, $b(t)$ must exist (Kubota, 1972) such that

$$P_1(t) = a^2(t) - b^2(t), \qquad P_2(t) = 2a(t)b(t), \qquad P_3(t) = a^2(t) + b^2(t).$$

Thus, any other rational adapted frame $(\mathbf{f}_1(t), \mathbf{f}_2(t), \mathbf{f}_3(t))$ on a PH curve can be expressed in terms of the ERF as $\mathbf{f}_1(t) = \mathbf{e}_1(t)$, and

$$\begin{aligned}
\mathbf{f}_2(t) &= \frac{a^2(t) - b^2(t)}{a^2(t) + b^2(t)}\mathbf{e}_2(t) - \frac{2a(t)b(t)}{a^2(t) + b^2(t)}\mathbf{e}_3(t), \\
\mathbf{f}_3(t) &= \frac{2a(t)b(t)}{a^2(t) + b^2(t)}\mathbf{e}_2(t) + \frac{a^2(t) - b^2(t)}{a^2(t) + b^2(t)}\mathbf{e}_3(t),
\end{aligned} \tag{7}$$

where $a(t), b(t)$ are polynomials with $\gcd(a(t), b(t)) = \text{constant}$.

[2] Bold font symbols are used to denote both complex numbers and vectors in $\mathbb{R}^3$ — the meaning should be clear from the context.

Remark 1. The focus of this paper is on rotation-minimizing *adapted* frames (RMAFs), which incorporate the unit tangent $\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$ as one frame vector, and the rotation of the frame vectors in the *normal plane* (orthogonal to $\mathbf{t}$) is minimized. The rotation-minimizing *directed* frame (RMDF), studied in Farouki and Giannelli (2009), incorporates the unit polar vector $\mathbf{o} = \mathbf{r}/|\mathbf{r}|$ as one component, and minimizes the rotation of the frame vectors in the *image plane* (orthogonal to $\mathbf{o}$). The RMDF is of interest in camera orientation control and related problems, and in Farouki and Giannelli (2009) it was shown the theory of RMAFs carries over to RMDFs, applied to the *anti-hodograph* (indefinite integral) of the given curve $\mathbf{r}(t)$. Therefore we focus on the RMAF here, and for brevity we designate it by RMF.

3. Spatial PH curves with rational RMFs

For brevity, curves with *rational rotation-minimizing frames* are henceforth called RRMF curves — such curves must be PH curves, since only PH curves possess rational unit tangents. Note that straight lines and planar PH curves are trivially RRMF curves. Since we are interested in non-degenerate RRMF curves (i.e., true space curves) we quote the following results from Farouki et al. (2009a), that allow us to discount these degenerate cases for PH cubics and quintics.

Remark 2. A spatial PH cubic degenerates to a straight line if and only if the coefficients of the linear polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ satisfy $\boldsymbol{\alpha}_1 : \boldsymbol{\beta}_1 = \boldsymbol{\alpha}_0 : \boldsymbol{\beta}_0$, and to a plane curve other than a straight line if and only if, for some real λ and complex $\mathbf{z}$, we have $\boldsymbol{\alpha}_1 = \lambda\boldsymbol{\alpha}_0 - \mathbf{z}\overline{\boldsymbol{\beta}}_0$ and $\boldsymbol{\beta}_1 = \lambda\boldsymbol{\beta}_0 + \mathbf{z}\overline{\boldsymbol{\alpha}}_0$.

Remark 3. A spatial PH quintic degenerates to a straight line if and only if the coefficients of the quadratic polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ satisfy $\boldsymbol{\alpha}_2 : \boldsymbol{\beta}_2 = \boldsymbol{\alpha}_1 : \boldsymbol{\beta}_1 = \boldsymbol{\alpha}_0 : \boldsymbol{\beta}_0$, and to a plane curve other than a straight line if and only if we have $\boldsymbol{\alpha}_1 = \lambda_1\boldsymbol{\alpha}_0 - \mu_1\mathbf{z}\overline{\boldsymbol{\beta}}_0$, $\boldsymbol{\alpha}_2 = \lambda_2\boldsymbol{\alpha}_0 - \mu_2\mathbf{z}\overline{\boldsymbol{\beta}}_0$ and $\boldsymbol{\beta}_1 = \lambda_1\boldsymbol{\beta}_0 + \mu_1\mathbf{z}\overline{\boldsymbol{\alpha}}_0$, $\boldsymbol{\beta}_2 = \lambda_2\boldsymbol{\beta}_0 + \mu_2\mathbf{z}\overline{\boldsymbol{\alpha}}_0$ for some real $\lambda_1, \lambda_2, \mu_1, \mu_2$ and complex $\mathbf{z}$, provided that the hodograph (4) is primitive.

A sufficient and necessary condition for the existence of a rational RMF on a spatial PH curve has been derived by Han, in terms of the quaternion representation: see Theorem 5 and related discussion in Han (2008). In terms of the Hopf map representation, this condition can be phrased as follows.

Theorem 1. *A regular PH curve defined in terms of two complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ by the hodograph* (4) *has a rational RMF if and only if a complex polynomial $\mathbf{w}(t) = a(t) + \mathrm{i}b(t)$ exists, where $a(t)$ and $b(t)$ are real polynomials with* $\gcd(a(t), b(t)) = \text{constant}$, *such that*

$$\frac{\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}' - \overline{\boldsymbol{\alpha}}'\boldsymbol{\alpha} + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}' - \overline{\boldsymbol{\beta}}'\boldsymbol{\beta}}{\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha} + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}} = \frac{\overline{\mathbf{w}}\mathbf{w}' - \overline{\mathbf{w}}'\mathbf{w}}{\overline{\mathbf{w}}\mathbf{w}}. \tag{8}$$

Note that, since the numerators in (8) amount to $2\mathrm{i}\,\mathrm{Im}(\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}' + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}')$ and $2\mathrm{i}\,\mathrm{Im}(\overline{\mathbf{w}}\mathbf{w}')$ and the denominators to $|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2$ and $|\mathbf{w}|^2$, respectively, this is essentially a relation between two *real* rational functions.

Remark 4. The polynomial $\mathbf{w}(t)$ in (8), written in Bernstein form as

$$\mathbf{w}(t) = \sum_{j=0}^{m} \mathbf{w}_j \binom{m}{j} (1-t)^{m-j} t^j,$$

is assumed to be nominally of the same degree as $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ in (5). However, it may be that (8) is satisfied in cases where the numerator and denominator of the expressions on the left or right have a non-constant common factor, and in such cases the degree of $\mathbf{w}(t)$ may differ from that of $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$. The case $\deg(\mathbf{w}(t)) < m$ is of no concern, since a polynomial of degree $< m$ has a non-trivial (degree-elevated) representation in the degree-m Bernstein basis. In the case $\deg(\mathbf{w}(t)) > m$, we must have $\gcd(\overline{\mathbf{w}}\mathbf{w}' - \overline{\mathbf{w}}'\mathbf{w}, \overline{\mathbf{w}}\mathbf{w}) \neq \text{constant}$ — in this case, analyzed in Appendix A, $\mathbf{w}(t)$ must have multiple roots.

Remark 5. When $\mathbf{w}(t)$ is either a *real* polynomial or a constant, condition (8) implies that

$$\overline{\boldsymbol{\alpha}}(t)\boldsymbol{\alpha}'(t) - \overline{\boldsymbol{\alpha}}'(t)\boldsymbol{\alpha}(t) + \overline{\boldsymbol{\beta}}(t)\boldsymbol{\beta}'(t) - \overline{\boldsymbol{\beta}}'(t)\boldsymbol{\beta}(t) = 0. \tag{9}$$

If this condition holds, the angle $\phi(t)$ between the ERF and RMF is constant. Since computation of the RMF incurs an integration constant, we may regard (9) as the condition identifying coincidence of the RMF and ERF: a detailed analysis of this condition was presented by Choi and Han (2002).

Henceforth, we assume that the polynomials (5) satisfy $|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2 \neq 0$ and $|\boldsymbol{\alpha}_m|^2 + |\boldsymbol{\beta}_m|^2 \neq 0$, since otherwise $\mathbf{r}'(t) = \mathbf{0}$ at $t = 0$ or 1. The following result helps to simplify analysis of the RRMF condition (8).

Lemma 1. *If* (8) *is satisfied for given complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ by a complex polynomial $\mathbf{w}(t)$, it is also satisfied by $\mathbf{c}\mathbf{w}(t)$ for any constant $\mathbf{c} \neq 0$. Thus, without loss of generality, one may set $\mathbf{w}_0 = 1$ as the leading Bernstein coefficient of $\mathbf{w}(t)$.*

Proof. The rational function on the right in (8) is unchanged if we replace $\mathbf{w}(t)$ by $\mathbf{c}\mathbf{w}(t)$, for any $\mathbf{c} \neq 0$. Since we must have $|\mathbf{w}_0| \neq 0$ if $|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2 \neq 0$, we may substitute $\mathbf{c}\mathbf{w}(t)$ with $\mathbf{c} = 1/\mathbf{w}_0$ for $\mathbf{w}(t)$. □

Interpreting the complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$, $\mathbf{w}(t)$ as curves in the complex plane, the expressions $\bar{\boldsymbol{\alpha}}(t)\boldsymbol{\alpha}'(t) - \bar{\boldsymbol{\alpha}}'(t)\boldsymbol{\alpha}(t)$, $\bar{\boldsymbol{\beta}}(t)\boldsymbol{\beta}'(t) - \bar{\boldsymbol{\beta}}'(t)\boldsymbol{\beta}(t)$, $\bar{\mathbf{w}}(t)\mathbf{w}'(t) - \bar{\mathbf{w}}'(t)\mathbf{w}(t)$ in (8) have an intuitive geometrical meaning: they are proportional to the *areal speed* of these curves — i.e., the rate at which the polar vector from the origin to the points of each curve sweeps out area. This interpretation deserves further consideration, but at present we find a direct algebraic analysis of condition (8) more profitable.

4. Characterization of RRMF cubics

Using the quaternion representation of spatial PH curves, Han (2008) has shown that only degenerate (linear or planar) cubics have rational RMFs. Prior to analysing quintic RRMF curves, it is instructive to deduce this result from the Hopf map condition (8), using the form of $\mathbf{w}(t)$ defined in Lemma 1. PH cubics are generated by choosing linear polynomials $\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}_0(1-t) + \boldsymbol{\alpha}_1 t$, $\boldsymbol{\beta}(t) = \boldsymbol{\beta}_0(1-t) + \boldsymbol{\beta}_1 t$ in (4). We assume they are relatively prime, otherwise the PH cubic degenerates to a straight line (see Remark 2). This implies that $\boldsymbol{\alpha}_0 : \boldsymbol{\alpha}_1 \neq \boldsymbol{\beta}_0 : \boldsymbol{\beta}_1$, and in particular $(\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0) \neq (0,0)$ and $(\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1) \neq (0,0)$.

Proposition 1. *A PH cubic defined by the Hopf map form* (4) *has a rational rotation-minimizing frame if and only if the Bernstein coefficients* $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1$ *and* $\boldsymbol{\beta}_0, \boldsymbol{\beta}_1$ *of the linear complex polynomials* $\boldsymbol{\alpha}(t)$ *and* $\boldsymbol{\beta}(t)$ *satisfy the constraint*

$$|\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1|^2 = \big(|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2\big)\big(|\boldsymbol{\alpha}_1|^2 + |\boldsymbol{\beta}_1|^2\big). \tag{10}$$

Proof. Han (2008) has shown that, in this case, condition (8) cannot be satisfied with $\deg(\mathbf{w}(t)) > 1$, so we may set $\mathbf{w}(t) = \mathbf{w}_0(1-t) + \mathbf{w}_1 t$. Comparing the numerators and denominators on the left and right of (8), we must have

$$\begin{aligned}
&\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 - \bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_0 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1 - \bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_0 = \gamma(\bar{\mathbf{w}}_0\mathbf{w}_1 - \bar{\mathbf{w}}_1\mathbf{w}_0),\\
&\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_0 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_0 = \gamma\bar{\mathbf{w}}_0\mathbf{w}_0,\\
&\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_0 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1 + \bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_0 = \gamma(\bar{\mathbf{w}}_0\mathbf{w}_1 + \bar{\mathbf{w}}_1\mathbf{w}_0),\\
&\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_1 = \gamma\bar{\mathbf{w}}_1\mathbf{w}_1,
\end{aligned}$$

for some non-zero real number γ. These four equations are equivalent to

$$\begin{aligned}
&\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_0 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_0 = \gamma\bar{\mathbf{w}}_0\mathbf{w}_0,\\
&\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1 = \gamma\bar{\mathbf{w}}_0\mathbf{w}_1,\\
&\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_1 = \gamma\bar{\mathbf{w}}_1\mathbf{w}_1.
\end{aligned} \tag{11}$$

By Lemma 1, we may take $\mathbf{w}_0 = 1$. The first two of Eqs. (11) then give

$$\gamma = |\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2, \qquad \mathbf{w}_1 = \frac{\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1}{|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2}. \tag{12}$$

To define a solution of Eqs. (11), these expressions for γ, $\mathbf{w}_0$, $\mathbf{w}_1$ must be compatible with the third equation. Substituting for γ, $\mathbf{w}_0$, $\mathbf{w}_1$ into this equation, and clearing denominators, yields the constraint (10). □

One can easily see that, when condition (10) is satisfied, the PH cubic $\mathbf{r}(t)$ degenerates to a straight line (whose RMF is trivially rational) — condition (10) is equivalent to $|\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0|^2 = 0$, so the linear polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ are proportional. By Remark 2, this situation identifies a straight line as a degenerate PH cubic. Now Proposition 1 treats the generic case, in which the left- and right-hand sides of (8) are not both identically zero. We address separately the special case (see Remark 5) in which both sides of (8) vanish.

Corollary 1. *When* $\mathrm{Im}(\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1) = 0$, *the polynomial* $\mathbf{w}(t)$ *is real, and* $\mathbf{r}(t)$ *degenerates to a planar PH cubic whose RMF is trivially rational.*

Proof. If $\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1$ is real, $\mathbf{w}(t) = \mathbf{w}_0(1-t) + \mathbf{w}_1 t$ is a real polynomial. Since $|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2 \neq 0$, we can write $\boldsymbol{\alpha}_1$, $\boldsymbol{\beta}_1$ in terms of complex numbers $\mathbf{c}$, $\mathbf{z}$ as $\boldsymbol{\alpha}_1 = \mathbf{c}\boldsymbol{\alpha}_0 - \mathbf{z}\bar{\boldsymbol{\beta}}_0$, $\boldsymbol{\beta}_1 = \mathbf{c}\boldsymbol{\beta}_0 + \mathbf{z}\bar{\boldsymbol{\alpha}}_0$. Then $\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1 = \mathbf{c}(|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2)$ has no imaginary part if and only if $\mathbf{c} = \lambda \in \mathbb{R}$. By Remark 2, this identifies a planar PH cubic (not a straight line), whose RMF is trivially rational. □

5. Characterization of RRMF quintics

Since no true spatial cubics possess rational RMFs, we now focus on quintics. Although the analysis is more involved, invoking Lemma 1 allows us to reduce the RRMF condition (8) to two simple algebraic constraints on the Bernstein coefficients of the quadratic polynomials $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$, that are sufficient and necessary for a rational RMF. Moreover, we show that the constraints admit solutions for the coefficients $\boldsymbol{\alpha}_1$, $\boldsymbol{\beta}_1$ (with one scalar freedom), for arbitrary choices of the coefficients $\boldsymbol{\alpha}_0$, $\boldsymbol{\beta}_0$, $\boldsymbol{\alpha}_2$, $\boldsymbol{\beta}_2$. An algorithm to construct RRMF quintics is formulated, that should be adaptable to meet geometric design requirements, and illustrative computed examples are included.

Since $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ are quadratic for PH quintics, $\bar{\boldsymbol{\alpha}}(t)\boldsymbol{\alpha}'(t)-\bar{\boldsymbol{\alpha}}'(t)\boldsymbol{\alpha}(t)+\bar{\boldsymbol{\beta}}(t)\boldsymbol{\beta}'(t)-\bar{\boldsymbol{\beta}}'(t)\boldsymbol{\beta}(t)$ is the quadratic polynomial

$$\begin{aligned}&2(\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1-\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_0+\bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1-\bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_0)(1-t)^2\\ &\quad+(\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_2-\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_0+\bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2-\bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_0)2(1-t)t\\ &\quad+2(\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2-\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_1+\bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2-\bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_1)t^2,\end{aligned}$$

and $\bar{\boldsymbol{\alpha}}(t)\boldsymbol{\alpha}(t)+\bar{\boldsymbol{\beta}}(t)\boldsymbol{\beta}(t)$ is the quartic polynomial

$$\begin{aligned}&(\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_0+\bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_0)(1-t)^4\\ &\quad+\tfrac{1}{2}(\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1+\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_0+\bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1+\bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_0)4(1-t)^3t\\ &\quad+\left[\tfrac{1}{6}(\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_2+\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_0+\bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2+\bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_0)+\tfrac{2}{3}(\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_1+\bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_1)\right]6(1-t)^2t^2\\ &\quad+\tfrac{1}{2}(\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2+\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_1+\bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2+\bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_1)4(1-t)t^3\\ &\quad+(\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_2+\bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_2)t^4.\end{aligned}$$

These forms are used to derive constraints on the coefficients of $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ that are sufficient and necessary for the satisfaction of (8) by some complex polynomial $\mathbf{w}(t)$, and hence the existence of a rational RMF.

Proposition 2. *A PH quintic specified by the Hopf map form* (4) *satisfies the rational rotation-minimizing frame condition* (8) *for some quadratic complex polynomial* $\mathbf{w}(t)$ *if and only if the coefficients* $\boldsymbol{\alpha}_0,\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2$ *and* $\boldsymbol{\beta}_0$, $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ *of the quadratic complex polynomials* $\boldsymbol{\alpha}(t)$ *and* $\boldsymbol{\beta}(t)$ *satisfy the constraint*

$$\big(|\boldsymbol{\alpha}_0|^2+|\boldsymbol{\beta}_0|^2\big)|\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2+\bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2|^2=\big(|\boldsymbol{\alpha}_2|^2+|\boldsymbol{\beta}_2|^2\big)|\boldsymbol{\alpha}_0\bar{\boldsymbol{\alpha}}_1+\boldsymbol{\beta}_0\bar{\boldsymbol{\beta}}_1|^2, \tag{13}$$

and either of the two constraints

$$\boldsymbol{\alpha}_0\boldsymbol{\beta}_1-\boldsymbol{\alpha}_1\boldsymbol{\beta}_0=0, \tag{14}$$

$$\big(|\boldsymbol{\alpha}_0|^2+|\boldsymbol{\beta}_0|^2\big)(\boldsymbol{\alpha}_0\boldsymbol{\beta}_2-\boldsymbol{\alpha}_2\boldsymbol{\beta}_0)=2(\boldsymbol{\alpha}_0\bar{\boldsymbol{\alpha}}_1+\boldsymbol{\beta}_0\bar{\boldsymbol{\beta}}_1)(\boldsymbol{\alpha}_0\boldsymbol{\beta}_1-\boldsymbol{\alpha}_1\boldsymbol{\beta}_0). \tag{15}$$

Proof. If $\mathbf{w}_0,\mathbf{w}_1,\mathbf{w}_2$ are the Bernstein coefficients of $\mathbf{w}(t)$, satisfaction of (8) implies that, for some non-zero real number γ, we have

$$\begin{aligned}&\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1-\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_0+\bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1-\bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_0=\gamma(\bar{\mathbf{w}}_0\mathbf{w}_1-\bar{\mathbf{w}}_1\mathbf{w}_0),\\ &\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_2-\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_0+\bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2-\bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_0=\gamma(\bar{\mathbf{w}}_0\mathbf{w}_2-\bar{\mathbf{w}}_2\mathbf{w}_0),\\ &\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2-\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_1+\bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2-\bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_1=\gamma(\bar{\mathbf{w}}_1\mathbf{w}_2-\bar{\mathbf{w}}_2\mathbf{w}_1),\\ &\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_0+\bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_0=\gamma\bar{\mathbf{w}}_0\mathbf{w}_0,\\ &\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1+\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_0+\bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1+\bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_0=\gamma(\bar{\mathbf{w}}_0\mathbf{w}_1+\bar{\mathbf{w}}_1\mathbf{w}_0),\\ &\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_2+\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_0+\bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2+\bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_0+4(\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_1+\bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_1)=\gamma(\bar{\mathbf{w}}_0\mathbf{w}_2+\bar{\mathbf{w}}_2\mathbf{w}_0+4\bar{\mathbf{w}}_1\mathbf{w}_1),\\ &\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2+\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_1+\bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2+\bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_1=\gamma(\bar{\mathbf{w}}_1\mathbf{w}_2+\bar{\mathbf{w}}_2\mathbf{w}_1),\\ &\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_2+\bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_2=\gamma\bar{\mathbf{w}}_2\mathbf{w}_2.\end{aligned}$$

These eight equations can be reduced to

$$\begin{aligned}&\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_0+\bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_0=\gamma\bar{\mathbf{w}}_0\mathbf{w}_0,\\ &\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1+\bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1=\gamma\bar{\mathbf{w}}_0\mathbf{w}_1,\\ &\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_2+\bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2+2(\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_1+\bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_1)=\gamma(\bar{\mathbf{w}}_0\mathbf{w}_2+2\bar{\mathbf{w}}_1\mathbf{w}_1),\\ &\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2+\bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2=\gamma\bar{\mathbf{w}}_1\mathbf{w}_2,\\ &\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_2+\bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_2=\gamma\bar{\mathbf{w}}_2\mathbf{w}_2.\end{aligned} \tag{16}$$

Now by Lemma 1, we may assume $\mathbf{w}_0 = 1$. The first of Eqs. (16) gives the proportionality constant

$$\gamma = |\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2,$$

and from the second equation we obtain

$$\mathbf{w}_1 = \frac{\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1}{|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2}. \tag{17}$$

Substituting γ, $\mathbf{w}_0$, $\mathbf{w}_1$ into the fourth equation then yields

$$\mathbf{w}_2 = \frac{\overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2}{\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_1 + \boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_1}. \tag{18}$$

To constitute a solution of the system (16), these expressions for $\gamma, \mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ must also satisfy the third and fifth of these equations. Substituting γ and $\mathbf{w}_2$ into the fifth equation and clearing denominators leads directly to condition (13). Similarly, upon substituting γ, $\mathbf{w}_0$, $\mathbf{w}_1$, $\mathbf{w}_2$ into the third equation and simplifying, we obtain

$$\begin{aligned}&\big(|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2\big)^2(\overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2) + 2|\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1|^2(\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_1 + \boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_1)\\ &\quad = \big(|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2\big)(\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_1 + \boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_1)\big[\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2 + 2\big(|\boldsymbol{\alpha}_1|^2 + |\boldsymbol{\beta}_1|^2\big)\big].\end{aligned}$$

By straightforward but laborious manipulations, this can be reduced to

$$(\overline{\boldsymbol{\alpha}}_0\overline{\boldsymbol{\beta}}_1 - \overline{\boldsymbol{\alpha}}_1\overline{\boldsymbol{\beta}}_0)\big[\big(|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2\big)(\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0) - 2(\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_1 + \boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_1)(\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0)\big] = 0.$$

To satisfy this condition, one of the factors on the left must vanish: constraint (14) corresponds to (the conjugate of) the first factor, and constraint (15) to the second factor. □

To obtain (18) we tacitly assumed that $\mathbf{w}_1 \neq 0$, i.e., $\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1 \neq 0$. If $\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1 = 0$, then $\mathbf{w}_1 = 0$ from the second of Eqs. (16), and hence $\overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2 = 0$ from the fourth. We now address this singular case.

Remark 6. Consider Eqs. (16) when $\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1 = \overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2 = 0$, and hence $\mathbf{w}_1 = 0$. Then the constraint (13) is evidently satisfied. Regarding $\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1 = \overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2 = 0$ as simultaneous equations for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\beta}_1$, we must have either $\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0 = 0$ or $\boldsymbol{\alpha}_1 = \boldsymbol{\beta}_1 = 0$. So either (14) or (15) is also satisfied. Eqs. (16) reduce in this case to

$$\begin{aligned}&\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_0 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_0 = \gamma\overline{\mathbf{w}}_0\mathbf{w}_0,\\ &\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2 + 2(\overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_1 + \overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_1) = \gamma\overline{\mathbf{w}}_0\mathbf{w}_2,\\ &\overline{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_2\boldsymbol{\beta}_2 = \gamma\overline{\mathbf{w}}_2\mathbf{w}_2.\end{aligned}$$

With $\mathbf{w}_0 = 1$, we have $\gamma = |\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2$ from the first equation, and

$$\mathbf{w}_2 = \frac{\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2 + 2(|\boldsymbol{\alpha}_1|^2 + |\boldsymbol{\beta}_1|^2)}{|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2}$$

from the second equation. Substituting γ, $\mathbf{w}_0$, $\mathbf{w}_2$ into the third equation and simplifying then yields the single constraint

$$\big(|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2\big)\big(|\boldsymbol{\alpha}_2|^2 + |\boldsymbol{\beta}_2|^2\big) = \big|\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2 + 2\big(|\boldsymbol{\alpha}_1|^2 + |\boldsymbol{\beta}_1|^2\big)\big|^2$$

in lieu of (13) and (14) or (15), when $\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1 = \overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2 = 0$.

Corollary 2. *When condition* (13) *is satisfied in conjunction with* (14)*, the PH quintic* $\mathbf{r}(t)$ *degenerates to a straight line, whose RMF is trivially rational.*

Proof. From condition (14) we must have $\boldsymbol{\beta}_0 = \mathbf{z}\boldsymbol{\alpha}_0$ and $\boldsymbol{\beta}_1 = \mathbf{z}\boldsymbol{\alpha}_1$ for some complex number $\mathbf{z}$. Substituting into (13), a laborious but straightforward calculation yields $|\mathbf{z}\boldsymbol{\alpha}_2 - \boldsymbol{\beta}_2|^2 = 0$, and hence $\boldsymbol{\beta}_2 = \mathbf{z}\boldsymbol{\alpha}_2$. Therefore, (13) and (14) imply that $\boldsymbol{\alpha}_2 : \boldsymbol{\beta}_2 = \boldsymbol{\alpha}_1 : \boldsymbol{\beta}_1 = \boldsymbol{\alpha}_0 : \boldsymbol{\beta}_0$, and we infer from Remark 3 that the curve must be a straight line. □

As in the cubic case, we address separately the special case in which (8) is satisfied with both sides vanishing identically (see Remark 5).

Corollary 3. *If* $\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1) = \mathrm{Im}(\overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2) = 0$*, the polynomial* $\mathbf{w}(t)$ *is real, and* $\mathbf{r}(t)$ *is a planar PH quintic whose RMF is trivially rational.*

Proof. When $\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1$ and $\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2 + \bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2$ are both real, the coefficients (17) and (18) are real, so $\mathbf{w}(t) = \mathbf{w}_0(1-t)^2 + \mathbf{w}_1 2(1-t)t + \mathbf{w}_2 t^2$ is a real polynomial. In this case, the third of Eqs. (16) implies that $\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_2 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2$ is also real. Since $|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2 \neq 0$ and $|\boldsymbol{\alpha}_2|^2 + |\boldsymbol{\beta}_2|^2 \neq 0$, we can invoke the argument used in Corollary 1 to write

$$\boldsymbol{\alpha}_1 = \lambda_1\boldsymbol{\alpha}_0 - \mathbf{z}_1\bar{\boldsymbol{\beta}}_0, \qquad \boldsymbol{\beta}_1 = \lambda_1\boldsymbol{\beta}_0 + \mathbf{z}_1\bar{\boldsymbol{\alpha}}_0, \tag{19}$$

$$\boldsymbol{\alpha}_2 = \lambda_2\boldsymbol{\alpha}_0 - \mathbf{z}_2\bar{\boldsymbol{\beta}}_0, \qquad \boldsymbol{\beta}_2 = \lambda_2\boldsymbol{\beta}_0 + \mathbf{z}_2\bar{\boldsymbol{\alpha}}_0, \tag{20}$$

$$\boldsymbol{\alpha}_1 = \lambda_3\boldsymbol{\alpha}_2 - \mathbf{z}_3\bar{\boldsymbol{\beta}}_2, \qquad \boldsymbol{\beta}_1 = \lambda_3\boldsymbol{\beta}_2 + \mathbf{z}_3\bar{\boldsymbol{\alpha}}_2, \tag{21}$$

for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \in \mathbb{C}$. Substituting from (20) for $\boldsymbol{\alpha}_2$, $\boldsymbol{\beta}_2$ into (21) and equating with (19) then gives

$$\lambda_3\lambda_2 - \mathbf{z}_3\bar{\mathbf{z}}_2 = \lambda_1, \qquad \lambda_3\mathbf{z}_2 + \lambda_2\mathbf{z}_3 = \mathbf{z}_1.$$

From the first equation, $\mathbf{z}_3\bar{\mathbf{z}}_2$ must be real. By writing $\mathbf{z}_2 = |\mathbf{z}_2|\exp(\mathrm{i}\phi_2)$ and $\mathbf{z}_3 = |\mathbf{z}_3|\exp(\mathrm{i}\phi_3)$, we have $\mathbf{z}_3\bar{\mathbf{z}}_2 = |\mathbf{z}_3||\mathbf{z}_2|\exp(\mathrm{i}(\phi_3 - \phi_2))$, so $\mathbf{z}_3\bar{\mathbf{z}}_2$ is real if and only if $\phi_2 = \phi_3 + k\pi$ for integer k, i.e., $\mathbf{z}_2 = c\mathbf{z}_3$ with $c \in \mathbb{R}$. Thus, writing $\mathbf{z}_3 = \mu_3\mathbf{z}$ with $\mathbf{z} \in \mathbb{C}$ and $\mu_3 \in \mathbb{R}$, we have $\mathbf{z}_2 = \mu_2\mathbf{z}$ with $\mu_2 = c\mu_3 \in \mathbb{R}$, and the second equation then gives $\mathbf{z} = \mu_1\mathbf{z}$ where $\mu_1 = \lambda_3\mu_2 + \lambda_2\mu_3 \in \mathbb{R}$. Hence, we can replace $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$ in (19)–(21) by $\mu_1\mathbf{z}$, $\mu_2\mathbf{z}$, $\mu_3\mathbf{z}$, and the coefficients of $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ have the form identified in Remark 3 as specifying a planar PH quintic (other than a straight line), whose RMF is trivially rational. □

Note that the analysis of RRMF cubics and quintics yields the same γ, $\mathbf{w}_1$ values, since the first two equations in (11) and (16) are identical.

Remark 7. Proposition 2 leaves open the possibility that additional types of RRMF quintics, satisfying (8) with $\deg(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) = 2$ and $\deg(\mathbf{w}(t)) > 2$, may exist — see Remark 4 and Appendix A.

We now show how conditions (13) and (15) provide a simple algorithm for the construction of RRMF quintics. Note first that (13) is a scalar condition, while (15) is a condition on complex values. Hence, these conditions impose three scalar constraints on the twelve parameters in $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, $\boldsymbol{\beta}_0$, $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$. Consequently, if we freely assign four of these complex coefficients *a priori*, we expect the algorithm to exhibit one residual scalar freedom.

Proposition 3. *For any choice of the coefficients $\boldsymbol{\alpha}_0$, $\boldsymbol{\beta}_0$, $\boldsymbol{\alpha}_2$, $\boldsymbol{\beta}_2$ that satisfy $|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2 \neq 0$, $|\boldsymbol{\alpha}_2|^2 + |\boldsymbol{\beta}_2|^2 \neq 0$ the constraints* (13) *and* (15) *identifying non-degenerate RRMF quintics admit solutions, with one free parameter, for the remaining coefficients $\boldsymbol{\alpha}_1$, $\boldsymbol{\beta}_1$.*

Proof. From (13) we can write

$$\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1 = k\sqrt{|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2}\exp(\mathrm{i}\theta_0), \tag{22}$$

$$\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_1 = k\sqrt{|\boldsymbol{\alpha}_2|^2 + |\boldsymbol{\beta}_2|^2}\exp(\mathrm{i}\theta_2), \tag{23}$$

for real k, θ_0, θ_2. Solving these as simultaneous equations for $\boldsymbol{\alpha}_1$, $\boldsymbol{\beta}_1$ gives[3]

$$\boldsymbol{\alpha}_1 = k\frac{\sqrt{|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2}\bar{\boldsymbol{\beta}}_2\exp(\mathrm{i}\theta_0) - \sqrt{|\boldsymbol{\alpha}_2|^2 + |\boldsymbol{\beta}_2|^2}\bar{\boldsymbol{\beta}}_0\exp(\mathrm{i}\theta_2)}{\bar{\boldsymbol{\alpha}}_0\bar{\boldsymbol{\beta}}_2 - \bar{\boldsymbol{\alpha}}_2\bar{\boldsymbol{\beta}}_0}, \tag{24}$$

$$\boldsymbol{\beta}_1 = k\frac{\sqrt{|\boldsymbol{\alpha}_2|^2 + |\boldsymbol{\beta}_2|^2}\bar{\boldsymbol{\alpha}}_0\exp(\mathrm{i}\theta_2) - \sqrt{|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2}\bar{\boldsymbol{\alpha}}_2\exp(\mathrm{i}\theta_0)}{\bar{\boldsymbol{\alpha}}_0\bar{\boldsymbol{\beta}}_2 - \bar{\boldsymbol{\alpha}}_2\bar{\boldsymbol{\beta}}_0}. \tag{25}$$

Substituting from (22) for $\boldsymbol{\alpha}_0\bar{\boldsymbol{\alpha}}_1 + \boldsymbol{\beta}_0\bar{\boldsymbol{\beta}}_1$ into (15), and the above expressions for $\boldsymbol{\alpha}_1$, $\boldsymbol{\beta}_1$ into the term $\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0$, and simplifying, we obtain

$$|\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0|^2 = 2k^2\Big[\sqrt{\big(|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2\big)\big(|\boldsymbol{\alpha}_2|^2 + |\boldsymbol{\beta}_2|^2\big)}\exp(\mathrm{i}\theta) - (\boldsymbol{\alpha}_0\bar{\boldsymbol{\alpha}}_2 + \boldsymbol{\beta}_0\bar{\boldsymbol{\beta}}_2)\Big] \tag{26}$$

where we define $\theta = \theta_2 - \theta_0$. Since the term on the left is real, the imaginary part of the term on the right must vanish — i.e., θ must be defined by

$$\sin\theta = \frac{\operatorname{Im}(\boldsymbol{\alpha}_0\bar{\boldsymbol{\alpha}}_2 + \boldsymbol{\beta}_0\bar{\boldsymbol{\beta}}_2)}{\sqrt{(|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2)(|\boldsymbol{\alpha}_2|^2 + |\boldsymbol{\beta}_2|^2)}}. \tag{27}$$

[3] We assume that $\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0 \neq 0$. Otherwise, we must have either $\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0 = 0$ or $\boldsymbol{\alpha}_0\bar{\boldsymbol{\alpha}}_1 + \boldsymbol{\beta}_0\bar{\boldsymbol{\beta}}_1 = 0$ from (15). The former identifies degeneration to a straight line (see Corollary 2). For the latter, we also have $\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2 + \bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2 = 0$ by (13) — this corresponds to the singular case treated in Remark 6.

The expression on the right always defines a permissible $\sin\theta$ value, since

$$\big(|\boldsymbol{\alpha}_0|^2+|\boldsymbol{\beta}_0|^2\big)\big(|\boldsymbol{\alpha}_2|^2+|\boldsymbol{\beta}_2|^2\big)=|\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_2+\boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_2|^2+|\boldsymbol{\alpha}_0\boldsymbol{\beta}_2-\boldsymbol{\alpha}_2\boldsymbol{\beta}_0|^2 \tag{28}$$

and the expression on the right is certainly not less than $\mathrm{Im}^2(\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_2+\boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_2)$. Once θ has been computed in this manner, the corresponding value of k^2 can be found from (26) as

$$k^2=\frac{\frac{1}{2}|\boldsymbol{\alpha}_0\boldsymbol{\beta}_2-\boldsymbol{\alpha}_2\boldsymbol{\beta}_0|^2}{\sqrt{(|\boldsymbol{\alpha}_0|^2+|\boldsymbol{\beta}_0|^2)(|\boldsymbol{\alpha}_2|^2+|\boldsymbol{\beta}_2|^2)}\cos\theta-\mathrm{Re}(\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_2+\boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_2)}. \tag{29}$$

Using (27) and (28), and choosing $\cos\theta$ positive, this can be re-written as

$$k^2=\frac{\frac{1}{2}|\boldsymbol{\alpha}_0\boldsymbol{\beta}_2-\boldsymbol{\alpha}_2\boldsymbol{\beta}_0|^2}{\sqrt{|\boldsymbol{\alpha}_0\boldsymbol{\beta}_2-\boldsymbol{\alpha}_2\boldsymbol{\beta}_0|^2+\mathrm{Re}^2(\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_2+\boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_2)}-\mathrm{Re}(\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_2+\boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_2)}, \tag{30}$$

where the right-hand side is clearly non-negative. Choosing θ_0 freely, setting $\theta_2=\theta+\theta_0$ with θ obtained from (27), and computing k from (30), we can determine $\boldsymbol{\alpha}_1$ and $\boldsymbol{\beta}_1$ from (24) and (25). □

The method for constructing RRMF quintics may be summarized as follows.

Algorithm.

1. Choose complex values $\boldsymbol{\alpha}_0$, $\boldsymbol{\beta}_0$, $\boldsymbol{\alpha}_2$, $\boldsymbol{\beta}_2$ with $|\boldsymbol{\alpha}_0|^2+|\boldsymbol{\beta}_0|^2\neq 0$, $|\boldsymbol{\alpha}_2|^2+|\boldsymbol{\beta}_2|^2\neq 0$;
2. Determine θ from expression (27);
3. Determine k from expression (30);
4. Choose θ_0 freely, and set $\theta_2=\theta_0+\theta$;
5. Compute $\boldsymbol{\alpha}_1$ and $\boldsymbol{\beta}_1$ from (24) and (25);
6. Construct the hodograph (4) from $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$.

It should be possible to impose desired geometrical constraints on the RRMF quintic $\mathbf{r}(t)$ under construction when selecting input values $\boldsymbol{\alpha}_0,\boldsymbol{\beta}_0,\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2$ for this algorithm (and choosing the parameter θ_0). In the Hermite interpolation algorithm (Farouki et al., 2002) for spatial PH quintics, based on the quaternion form (3), the coefficients $\mathcal{A}_0=\boldsymbol{\alpha}_0+\mathbf{k}\boldsymbol{\beta}_0$ and $\mathcal{A}_2=\boldsymbol{\alpha}_2+\mathbf{k}\boldsymbol{\beta}_2$ of the quadratic quaternion polynomial $\mathcal{A}(t)$ are fixed (modulo one scalar freedom each) by interpolating the end-derivatives $\mathbf{r}'(0)$ and $\mathbf{r}'(1)$, while interpolation of the displacement $\mathbf{r}(1)-\mathbf{r}(0)$ determines $\mathcal{A}_1=\boldsymbol{\alpha}_1+\mathbf{k}\boldsymbol{\beta}_1$. It can be shown (Farouki et al., 2008) that, among the two-parameter family of interpolants, one parameter essentially controls the arc length while the other controls the curve shape at fixed arc length. Since the conditions (13) and (15) for an RRMF quintic amount to three scalar constraints, it will be necessary to relax from C^1 to G^1 Hermite data — i.e., interpolation of the end-tangents $\mathbf{t}(0)=\mathbf{r}'(0)/|\mathbf{r}'(0)|$ and $\mathbf{t}(1)=\mathbf{r}'(1)/|\mathbf{r}'(1)|$. However, a detailed treatment of this problem would incur an extraordinary digression from our present focus, and we defer it to another paper.

Example 1. Consider the choices

$$\boldsymbol{\alpha}_0=1+2\mathrm{i},\qquad \boldsymbol{\beta}_0=-2+\mathrm{i},\qquad \boldsymbol{\alpha}_2=2-\mathrm{i},\qquad \boldsymbol{\beta}_2=-1+2\mathrm{i},$$

for which $\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_2+\boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_2=4+8\mathrm{i}$, $\boldsymbol{\alpha}_0\boldsymbol{\beta}_2-\boldsymbol{\alpha}_2\boldsymbol{\beta}_0=-2-4\mathrm{i}$, and $|\boldsymbol{\alpha}_0|^2+|\boldsymbol{\beta}_0|^2=|\boldsymbol{\alpha}_2|^2+|\boldsymbol{\beta}_2|^2=10$. Then (27) and (30) give $\sin\theta=4/5$ and $k=\sqrt{5}$. Taking $\theta_0=0$ and $\theta_2=\theta$, we have $\exp(\mathrm{i}\theta_0)=1$ and $\exp(\mathrm{i}\theta_2)=(3+4\mathrm{i})/5$, and from (24) and (25) we obtain

$$\boldsymbol{\alpha}_1=\frac{1+\mathrm{i}}{\sqrt{2}}\quad\text{and}\quad \boldsymbol{\beta}_1=\frac{-3+\mathrm{i}}{\sqrt{2}}.$$

From (17) and (18), the coefficients of $\mathbf{w}(t)$ are determined to be

$$\mathbf{w}_0=1,\qquad \mathbf{w}_1=\frac{1}{\sqrt{2}},\qquad \mathbf{w}_2=\frac{3-4\mathrm{i}}{5},$$

and one can easily verify the complex quadratic polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$, $\mathbf{w}(t)$ defined by these coefficients satisfy (8).

For this example, the polynomials $a(t)=\mathrm{Re}(\mathbf{w}(t))$, $b(t)=\mathrm{Im}(\mathbf{w}(t))$ that define the rational rotation (7) of the ERF onto the RMF are given by

$$a(t)=(1-t)^2+\frac{1}{\sqrt{2}}2(1-t)t+\frac{3}{5}t^2,\qquad b(t)=-\frac{4}{5}t^2.$$

Once the Bernstein coefficients $\boldsymbol{\alpha}_0,\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2$ and $\boldsymbol{\beta}_0,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2$ of the two quadratic polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ are known, the ERF can be constructed from (6).

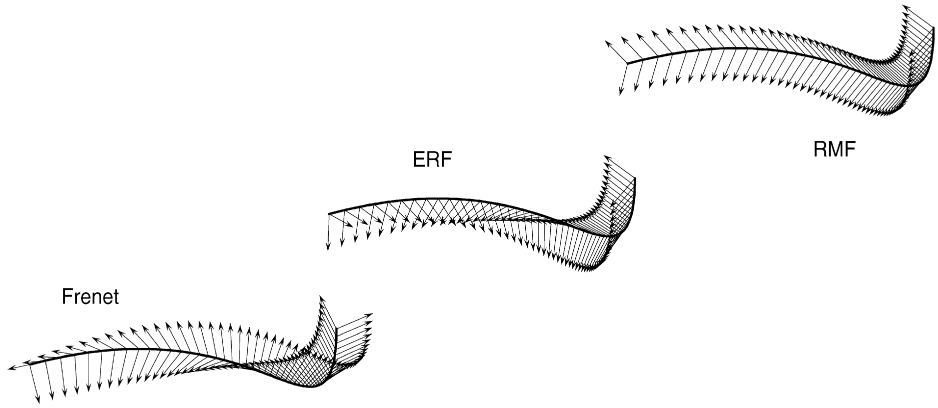

Fig. 1. The RRMF quintic of Example 1, showing the Frenet frame (left), Euler–Rodrigues frame (center), and the rotation-minimizing frame (right). For clarity, the unit tangent vector $\mathbf{t} = \mathbf{e}_1 = \mathbf{f}_1$ (common to all three adapted frames) is not shown — only the two normal-plane vectors are illustrated.

The ERF vectors $\mathbf{e}_1(t)$, $\mathbf{e}_2(t)$, $\mathbf{e}_3(t)$ have a rational quartic dependence on the curve parameter t. Since the polynomials $a(t)$, $b(t)$ in (7) are quadratic, the RMF vectors $\mathbf{f}_2(t)$, $\mathbf{f}_3(t)$ are nominally rational functions of degree 8 in t. Since the expressions for the ERF and RMF vectors are rather cumbersome, we refrain from quoting them here. The MAPLE computer algebra system was used to compute them, and to verify that the ω_1 component of the angular velocity $\boldsymbol{\omega}$, given by (1), vanishes.

To construct the Bézier form of the RRMF quintic defined by integrating (4), it is convenient to convert to the quaternion form (3). The quaternion coefficients $\mathcal{A}_r = \boldsymbol{\alpha}_r + \mathbf{k}\boldsymbol{\beta}_r$ for $r = 0, 1, 2$ of $\mathcal{A}(t)$ are

$$\mathcal{A}_0 = 1 + 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}, \qquad \mathcal{A}_1 = \frac{1 + \mathbf{i} + \mathbf{j} - 3\mathbf{k}}{\sqrt{2}}, \qquad \mathcal{A}_2 = 2 - \mathbf{i} + 2\mathbf{j} - \mathbf{k},$$

and in terms of them we have (Farouki et al., 2002) the control points

$$\begin{aligned}
\mathbf{p}_1 &= \mathbf{p}_0 + \frac{1}{5}\mathcal{A}_0\mathbf{i}\mathcal{A}_0^*, \\
\mathbf{p}_2 &= \mathbf{p}_1 + \frac{1}{10}(\mathcal{A}_0\mathbf{i}\mathcal{A}_1^* + \mathcal{A}_1\mathbf{i}\mathcal{A}_0^*), \\
\mathbf{p}_3 &= \mathbf{p}_2 + \frac{1}{30}(\mathcal{A}_0\mathbf{i}\mathcal{A}_2^* + 4\mathcal{A}_1\mathbf{i}\mathcal{A}_1^* + \mathcal{A}_2\mathbf{i}\mathcal{A}_0^*), \\
\mathbf{p}_4 &= \mathbf{p}_3 + \frac{1}{10}(\mathcal{A}_1\mathbf{i}\mathcal{A}_2^* + \mathcal{A}_2\mathbf{i}\mathcal{A}_1^*), \\
\mathbf{p}_5 &= \mathbf{p}_4 + \frac{1}{5}\mathcal{A}_2\mathbf{i}\mathcal{A}_2^*,
\end{aligned}$$

the initial control point $\mathbf{p}_0$ being an arbitrary integration constant. Fig. 1 illustrates the RRMF quintic, together with its ERF and RMF.

Although the RMF frame vectors $\mathbf{f}_2$, $\mathbf{f}_3$ and angular velocity components ω_2, ω_3 are rather complicated, the RMF angular velocity magnitude $|\boldsymbol{\omega}|$ has a fairly manageable expression, namely

$$\frac{\sqrt{8(13 + 8\sqrt{2})}}{\sqrt{82t^4 + (52\sqrt{2} - 100)t^3 + (118 - 22\sqrt{2})t^2 - (100 + 30\sqrt{2})t + 65 + 40\sqrt{2}}}.$$

For comparison, the angular velocity magnitude $|\boldsymbol{\omega}|$ for the ERF is

$$\frac{c\sqrt{(62t^2 - (14 - 6\sqrt{2})t + 8 + \sqrt{2})(14t^2 - (30 + 10\sqrt{2})t + 40 + 25\sqrt{2})}}{82t^4 + (52\sqrt{2} - 100)t^3 + (118 - 22\sqrt{2})t^2 - (100 + 30\sqrt{2})t + 65 + 40\sqrt{2}},$$

where $c = 2\sqrt{(1005 + 568\sqrt{2})/217}$. Fig. 2 compares these angular speeds.

Remark 8. The RRMF quintic conditions (13) and (15) can, in principle, be expressed in terms of the quaternion representation for PH curves, using (Farouki et al., 2009a) the conversion

$$\boldsymbol{\alpha}_r = \frac{1}{2}(\mathcal{A}_r - \mathbf{i}\mathcal{A}_r\mathbf{i}) \quad \text{and} \quad \boldsymbol{\beta}_r = -\frac{1}{2}\mathbf{k}(\mathcal{A}_r + \mathbf{i}\mathcal{A}_r\mathbf{i}) \tag{31}$$

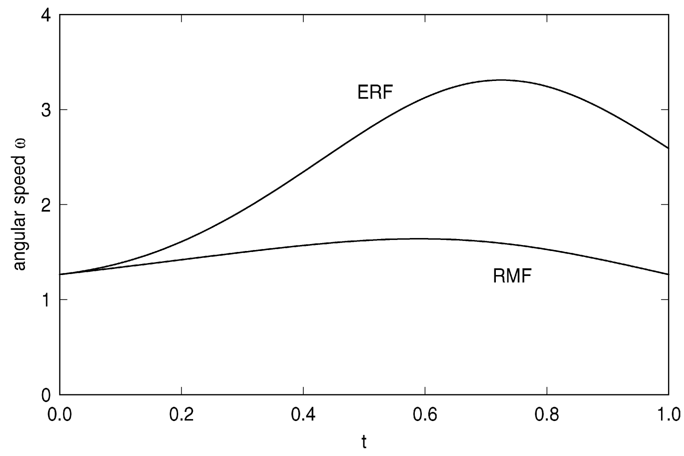

Fig. 2. Variation of angular velocity magnitude for the Euler–Rodrigues frame and rotation-minimizing frame, along the RRMF quintic of Example 1.

for $r = 0, 1, 2$ between the Hopf map and quaternion coefficients. Then we have $|\boldsymbol{\alpha}_r|^2 + |\boldsymbol{\beta}_r|^2 = |\mathcal{A}_r|^2$, but the terms $\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1$, $\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2 + \bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2$ and $\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0$, $\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0$ do not have simple and intuitive expressions in terms of $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$. Thus, the Hopf map form seems much better suited to the study of RRMF curves than the quaternion form.

6. Higher-order RRMF curves

The approach used in Propositions 1 and 2 to determine conditions on the coefficients of the polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ that are sufficient and necessary for rational RMFs on PH cubics and quintics can be extended to higher-order curves. To obtain RRMF curves of degree 7, for example, we must use cubic complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ and the system of equations analogous to (11) and (16) in the case of RRMF cubics and quintics becomes

$$\begin{aligned}
&\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_0 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_0 = \gamma\,\bar{\mathbf{w}}_0\mathbf{w}_0,\\
&\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1 = \gamma\,\bar{\mathbf{w}}_0\mathbf{w}_1,\\
&2(\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_2 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2) + 3(\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_1) = \gamma\,(2\bar{\mathbf{w}}_0\mathbf{w}_2 + 3\bar{\mathbf{w}}_1\mathbf{w}_1),\\
&\bar{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3 + \bar{\boldsymbol{\beta}}_0\boldsymbol{\beta}_3 + 6(\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2 + \bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2) + 3(\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_1) = \gamma\,(\bar{\mathbf{w}}_0\mathbf{w}_3 + 6\bar{\mathbf{w}}_1\mathbf{w}_2 + 3\bar{\mathbf{w}}_2\mathbf{w}_1),\\
&2(\bar{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_3 + \bar{\boldsymbol{\beta}}_1\boldsymbol{\beta}_3) + 3(\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_2 + \bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_2) = \gamma\,(2\bar{\mathbf{w}}_1\mathbf{w}_3 + 3\bar{\mathbf{w}}_2\mathbf{w}_2),\\
&\bar{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_3 + \bar{\boldsymbol{\beta}}_2\boldsymbol{\beta}_3 = \gamma\,\bar{\mathbf{w}}_2\mathbf{w}_3,\\
&\bar{\boldsymbol{\alpha}}_3\boldsymbol{\alpha}_3 + \bar{\boldsymbol{\beta}}_3\boldsymbol{\beta}_3 = \gamma\,\bar{\mathbf{w}}_3\mathbf{w}_3.
\end{aligned}$$

Taking $\mathbf{w}_0 = 1$ again, we have $\gamma = |\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2$ from the first equation, and from the second we see that $\mathbf{w}_1$ is given by the same expression (12) and (17) as in the cubic and quintic cases. Then $\mathbf{w}_2$ and $\mathbf{w}_3$ can be directly obtained in terms of the coefficients of the cubics $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ from the third and fourth equations. Substituting these expressions for $\gamma, \mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ into the fifth, sixth, and seventh equations then yields a set of constraints on the $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ coefficients that are sufficient and necessary for the degree 7 spatial PH curve specified by (4) to possess a rational RMF. Since the constraints in this case are rather involved, we shall not further tax the reader's endurance.

7. Closure

A method for constructing quintic curves with rational rotation-minimizing frames has been presented. These "RRMF quintics" — which are necessarily PH curves — constitute the lowest-order non-degenerate curves with rational RMFs. The construction is based upon the Hopf map form (4) of spatial PH curves, from which constraints on the coefficients of the complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ are derived that characterize the existence of rational RMFs on PH quintics. Using these constraints, a simple algorithm was formulated that computes suitable values for the $\boldsymbol{\alpha}_1$, $\boldsymbol{\beta}_1$ Bernstein coefficients of $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ when $\boldsymbol{\alpha}_0$, $\boldsymbol{\beta}_0$ and $\boldsymbol{\alpha}_2$, $\boldsymbol{\beta}_2$ have been specified *a priori*. The algorithm should be modifiable to permit geometric design using RRMF quintics, through a geometric Hermite interpolation scheme (this problem is deferred to another paper). The approach to characterizing RRMF curves presented herein also gives a simple demonstration of the known fact that all RRMF cubics are degenerate (straight lines or planar curves), and permits extensions to the characterization of RRMF curves of degree 7 or higher.

Appendix A. Analysis of $\gcd(\overline{\mathbf{w}}\mathbf{w}' - \overline{\mathbf{w}}'\mathbf{w}, \overline{\mathbf{w}}\mathbf{w})$

We consider here the possibility that the numerator and denominator of the expression on the right in (8) possess a non-constant common factor, that can be cancelled out. Suppose $\mathbf{w}(t) = a(t) + \mathrm{i}b(t)$, where $\gcd(a(t), b(t)) = \text{constant}$, has r distinct roots $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_r$ with multiplicities $m_1, \ldots, m_r$ so that $m_1 + \cdots + m_r = \deg(\mathbf{w}(t))$. Then for some complex constant $\mathbf{c}$, we have

$$\mathbf{w}(t) = \mathbf{c} \prod_{j=1}^{r} (t - \boldsymbol{\mu}_j)^{m_j}. \tag{32}$$

Note that none of $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_r$ can be real or complex conjugates, since such roots contradict the condition $\gcd(a(t), b(t)) = \text{constant}$.

From (32), the denominator of the expression on the right in (8) is

$$\overline{\mathbf{w}}(t)\mathbf{w}(t) = |\mathbf{c}|^2 \prod_{j=1}^{r} \big[(t - \boldsymbol{\mu}_j)(t - \overline{\boldsymbol{\mu}}_j)\big]^{m_j}, \tag{33}$$

and writing the derivative of $\mathbf{w}(t)$ as

$$\mathbf{w}'(t) = \mathbf{w}(t) \sum_{k=1}^{r} \frac{m_k}{t - \boldsymbol{\mu}_k},$$

the numerator of this expression can be written as

$$\begin{aligned}\overline{\mathbf{w}}(t)\mathbf{w}'(t) - \mathbf{w}(t)\overline{\mathbf{w}}'(t) &= \overline{\mathbf{w}}(t)\mathbf{w}(t) \sum_{k=1}^{r} \frac{m_k}{t - \boldsymbol{\mu}_k} - \frac{m_k}{t - \overline{\boldsymbol{\mu}}_k} \\ &= |\mathbf{c}|^2 \sum_{k=1}^{r} m_k(\boldsymbol{\mu}_k - \overline{\boldsymbol{\mu}}_k) \prod_{j=1}^{r} \big[(t - \boldsymbol{\mu}_j)(t - \overline{\boldsymbol{\mu}}_j)\big]^{m_j - \delta_{jk}},\end{aligned} \tag{34}$$

where we use the Kronecker delta

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Comparing (33) and (34), common factors of these complex polynomials may be identified. The polynomial (33) is the product of terms $[(t - \boldsymbol{\mu}_j)(t - \overline{\boldsymbol{\mu}}_j)]^{m_j}$ for $j = 1, \ldots, r$. Obviously, its roots are simply the roots $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_r$ of $\mathbf{w}(t)$, together with their conjugates $\overline{\boldsymbol{\mu}}_1, \ldots, \overline{\boldsymbol{\mu}}_r$, and $\boldsymbol{\mu}_j$, $\overline{\boldsymbol{\mu}}_j$ have multiplicity m_j as roots of (33). On the other hand, the polynomial (34) is a sum of products of the factors $[(t - \boldsymbol{\mu}_j)(t - \overline{\boldsymbol{\mu}}_j)]^{m_j}$ for $j = 1, \ldots, r$ — but with the exponent of $[(t - \boldsymbol{\mu}_k)(t - \overline{\boldsymbol{\mu}}_k)]$ reduced by 1 in the kth term for $k = 1, \ldots, r$.

From (33) and (34) it is clear that, for these two polynomials to possess a common root, at least one of $m_1, \ldots, m_r$ must be greater than 1, i.e., $\mathbf{w}(t)$ must have at least one multiple root. Writing

$$\mathbf{h}(t) = \gcd\big(\overline{\mathbf{w}}(t)\mathbf{w}'(t) - \mathbf{w}(t)\overline{\mathbf{w}}'(t), \overline{\mathbf{w}}(t)\mathbf{w}(t)\big),$$

so that $\overline{\mathbf{w}}(t)\mathbf{w}'(t) - \mathbf{w}(t)\overline{\mathbf{w}}'(t) = \mathbf{h}(t)\mathbf{p}(t)$, $\overline{\mathbf{w}}(t)\mathbf{w}(t) = \mathbf{h}(t)\mathbf{q}(t)$, and hence

$$\frac{\overline{\mathbf{w}}(t)\mathbf{w}'(t) - \overline{\mathbf{w}}'(t)\mathbf{w}(t)}{\overline{\mathbf{w}}(t)\mathbf{w}(t)} = \frac{\mathbf{p}(t)}{\mathbf{q}(t)} \tag{35}$$

for relatively prime complex polynomials $\mathbf{p}(t)$ and $\mathbf{q}(t)$, one can easily verify from (33) and (34) that, up to a (complex) constant factor,

$$\mathbf{h}(t) = \prod_{j=1}^{r} \big[(t - \boldsymbol{\mu}_j)(t - \overline{\boldsymbol{\mu}}_j)\big]^{m_j - 1}. \tag{36}$$

Clearly, $\mathbf{h}(t)$ is actually a *real* polynomial. Comparing (33) and (34) with (36), we see that

$$\mathbf{p}(t) = 2\mathrm{i}|\mathbf{c}|^2 \sum_{k=1}^{r} m_k \operatorname{Im}(\boldsymbol{\mu}_k) \prod_{j=1}^{r} \big[(t - \boldsymbol{\mu}_j)(t - \overline{\boldsymbol{\mu}}_j)\big]^{1 - \delta_{jk}}, \tag{37}$$

$$\mathbf{q}(t) = |\mathbf{c}|^2 \prod_{j=1}^{r} \big[(t - \boldsymbol{\mu}_j)(t - \overline{\boldsymbol{\mu}}_j)\big]. \tag{38}$$

If $d = m_1 + \cdots + m_r = \deg(\mathbf{w}(t))$, then $\deg(\overline{\mathbf{w}}(t)\mathbf{w}'(t) - \mathbf{w}(t)\overline{\mathbf{w}}'(t)) = 2d - 2$ and $\deg(\overline{\mathbf{w}}(t)\mathbf{w}(t)) = 2d$, and the degree of (36) is

$$\ell = \sum_{j=1}^{r} 2(m_j - 1).$$

Consequently, we have $\deg(\mathbf{p}(t)) = 2d - 2 - \ell$ and $\deg(\mathbf{q}(t)) = 2d - \ell$ in the reduced form (35).

Remark 9. Up to a constant, the common factor of $\overline{\mathbf{w}}(t)\mathbf{w}'(t) - \mathbf{w}(t)\overline{\mathbf{w}}'(t)$ and $\overline{\mathbf{w}}(t)\mathbf{w}(t)$ given by (36) is simply $\mathbf{f}(t)\bar{\mathbf{f}}(t)$, where $\mathbf{f}(t) = \gcd(\mathbf{w}(t), \mathbf{w}'(t))$. However, if $\mathbf{w}(t)$ satisfies (8) for given $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ the “reduced” or *square-free* polynomial $\mathbf{w}_r(t) = \mathbf{w}(t)/\mathbf{f}(t)$ — which has each root $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_r$ of $\mathbf{w}(t)$ as a *simple* root — does not satisfy (8) if any of the root multiplicities $m_1, \ldots, m_r$ exceed 1. Hence (8) might be satisfied by a polynomial $\mathbf{w}(t)$ of degree $d > 2$ with multiple roots, it is not satisfied by polynomials with only simple roots.

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