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# $C^1$-Continuity of the generalized four-point scheme

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ABSTRACT

We show that the interpolatory four-point-scheme with tension parameter $\omega$ generates $C^1$-limit curves if and only if $0 < \omega < \omega^*$, where $\omega^* \approx 0.19273$ is the unique real solution of the cubic equation $32\omega^3 + 4\omega - 1 = 0$.

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## 1. Introduction

The interpolatory four-point subdivision scheme was introduced in 1986 by Dubuc [5], and generalized in 1987 by means of a real tension parameter $\omega$ by Dyn and Levin [6,7]. Starting from initial data $c_0^k \in \mathbb{R}, k \in \mathbb{Z}$, new values at level $n+1 \in \mathbb{N}$ are generated by the recursion

$$c_{n+1}^{2k} = c_n^k,$$

$$c_{n+1}^{2k+1} = \left(\frac{1}{2} + \omega\right)\left(c_n^k + c_n^{k+1}\right) - \omega\left(c_n^{k-1} + c_n^{k+2}\right).$$

Pairing the values $c_n^k$ at level $n$ with the dyadic abscissae $x_n^k := k/2^n$ and connecting these points by straight lines, we obtain a piecewise linear function $c_n : \mathbb{R} \to \mathbb{R}$ with

$$c_n\left(x_n^k\right) = c_n^k.$$

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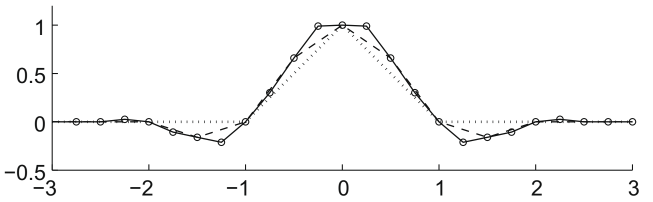


**Fig. 1.** Two steps of the four-point scheme for $\omega = 0.16$.

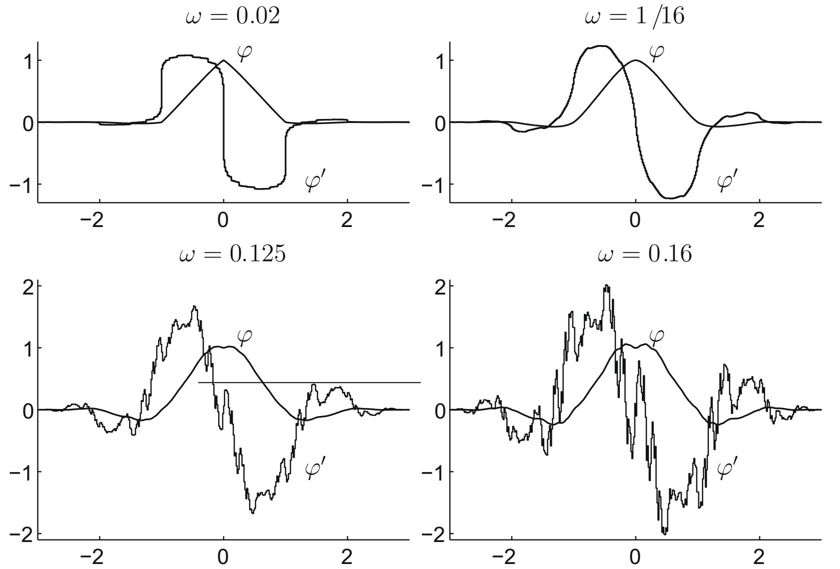


**Fig. 2.** Examples of nodal functions and their derivatives generated by the four-point scheme for different values of $\omega$.

The point-wise limit of this sequence of functions, if it exists, is denoted by

$$c := \lim_{n\to\infty} c_n.$$

The limit function $\varphi$ obtained for the initial data $c_0^k := \delta_{k,0}$ is compactly supported, and called the *nodal function* of the scheme. It admits the representations

$$c = \sum_{k\in\mathbb{Z}} c_0^k \varphi(\cdot - k)$$

so that, in general, $c$ is as smooth as $\varphi$. Fig. 1 shows the first two recursion steps of the four-point scheme for $c_0^k := \delta_{k,0}$ and $\omega = 0.16$. In Fig. 2, nodal functions and their derivatives are displayed for different values of $\omega$.

The question arises how the smoothness of $\varphi$ depends on the choice of $\omega$. In particular, the problem of determining the set

$$\Omega := \left\{\omega \in \mathbb{R} : \varphi \text{ is } C^1\right\}$$

of $C^1$-parameters has attracted some attention. Dubuc [5] proved $\varphi$ to be $C^1$ for his special choice $\omega = 1/16$. However, for general $\omega$, the problem is in fact quite subtle. Below, we briefly recall a few basic facts from subdivision analysis, and apply them to our special setting. Details can be found in [3,4,9,10].

The differences

$$d_n^k := 2^{-n}\left(c_n^{k+1} - c_n^k\right) - 2^{-n}\left(c_n^k - c_n^{k-1}\right), \quad k \in \mathbb{Z}$$

of consecutive slopes of the functions $c_n$ satisfy the recurrence relation

$$d_{n+1}^{2k} = (1-4\omega)\, d_n^k - 2\omega\left(d_n^{k-1} + d_n^{k+1}\right),$$
$$d_{n+1}^{2k+1} = 4\omega\left(d_n^k + d_n^{k+1}\right)$$

and the given subdivision scheme is $C^1$ if and only if

$$\lim_{n\to\infty}\sup_{k\in\mathbb{Z}} |d_n^k| = 0 \tag{1}$$

for any choice of initial data. There exist two formally different, but actually equivalent, approaches in assessing this condition. The first one [9] is based on considerations of the formal power series $\sum_{k\in\mathbb{Z}} d_n^k z^k$, while the second one [10] relies on the following matrix formalism: With $\omega' := 1 - 4\omega$, let

$$A_0^\omega := \begin{bmatrix} 4\omega & 4\omega & 0 & 0 \\ -2\omega & \omega' & -2\omega & 0 \\ 0 & 4\omega & 4\omega & 0 \\ 0 & -2\omega & \omega' & -2\omega \end{bmatrix}, \quad A_1^\omega := \begin{bmatrix} -2\omega & \omega' & -2\omega & 0 \\ 0 & 4\omega & 4\omega & 0 \\ 0 & -2\omega & \omega' & -2\omega \\ 0 & 0 & 4\omega & 4\omega \end{bmatrix}. \tag{2}$$

For each column-vector $D_n^k := \left(d_n^k, \ldots, d_n^{k+3}\right)$ of four consecutive differences of slopes at level $n$ there exists an index vector $I = (i_1, \ldots, i_n) \in \{0,1\}^n$ and a column-vector $D_0^\ell = \left(d_0^\ell, \ldots, d_0^{\ell+3}\right)$ of initial differences of slopes such that

$$D_n^k = A_{i_n}^\omega \cdots A_{i_1}^\omega D_0^\ell.$$

Thus, the values $d_n^k$ are related to the set of all possible products of the matrices $A_0^\omega, A_1^\omega$. A well-known criterion states that contractivity according to (1) is equivalent to find $n \in \mathbb{N}$ such that the row-sum norm

$$\|A_{i_n}^\omega \cdots A_{i_1}^\omega\| < 1 \tag{3}$$

for all index vectors $I \in \{0,1\}^n$ of length $n$, see [10].

In [7], $C^1$-limits were established for $0 < \omega < 1/8$ by considering all products of length $n = 2$, and ruled out for $\omega \geqslant 1/4$ and $\omega \leq 0$. Later on, in [8], the verified range of $C^1$-parameters was extended to $0 < \omega < \left(\sqrt{5} - 1\right)/8 \approx 0.154$ using $n = 3$. Increasing the length of products provides sharper results, and the brute-force-approach of Höfler [2] yielded the bound $\omega < 0.18653$ by considering all products of length $n = 22$. By computing the spectral radii of all products of length 2, Villemoes [11] proved that parameters $\omega$ greater than or equal to

$$\omega^* := \frac{1}{12}\left(27 + 3\sqrt{105}\right)^{1/3} - \frac{1}{2}\left(27 + 3\sqrt{105}\right)^{-1/3} \approx 0.19273$$

do not belong to the set $\Omega$ of $C^1$-parameters, and he conjectured $\Omega = (0, \omega^*)$. In this paper, we are going to confirm this claim.

## 2. The joint spectral radius

Let $\mathcal{A} = \{A_0, A_1\}, A_i \in \mathbb{R}^{d\times d}, d \in \mathbb{N}$, be a set of two real quadratic matrices. For a sequence of indices $I = (i_1, i_2, \ldots, i_n) \in \{0,1\}^n$ with length $|I| = n$ we define the product $A_I := A_{i_n} \cdots A_{i_2} A_{i_1}$. The asymptotic behavior of $A_I$ as $n \to \infty$ is closely related to the concept of the *joint spectral radius*. It is defined by

$$\varrho(\mathcal{A}) := \limsup_{n\to\infty} \sup_{|I|=n} \|A_I\|^{1/n},$$

where $\|\cdot\|$ denotes the row-sum norm. The matrix $A_I$ is called *contractive*, if $\|A_I\| < 1$, and the set $\mathcal{A}$ is called *contractive*, if $\varrho(\mathcal{A}) < 1$.

The set of all index vectors $\mathcal{I}:=\{I \in \{0,1\}^n : n \in \mathbb{N}_0\}$ can be regarded as a directed binary tree $\mathcal{T}:=(\mathcal{I},\mathcal{E})$ with knots $\mathcal{I}$ and edges

$$\mathcal{E}:=\left\{(I,J)\in\mathcal{I}\times\mathcal{I}: J=(I,0) \text{ or } J=(I,1)\right\}.$$

Let us introduce some standard terminology from graph theory: The *root* of the tree $\mathcal{T}$ is the empty set, and, by convention, the corresponding matrix product is the identity matrix $\mathrm{Id}:=A_{\emptyset}$. The knot $J$ is a *child* of $I$ if these knots are connected by an edge, i.e., $(I,J)\in\mathcal{E}$. If $J=(I,I')$, then $I$ is a *prefix* of $J$. We write

$$I \leqslant J \quad \text{and} \quad I \nleqslant J$$

to indicate that $I$ *is* or *is not* a prefix of $J$, respectively. A *subtree* $\mathcal{T}'=(\mathcal{I}',\mathcal{E}')$ of $\mathcal{T}$ is a tree with knots $\mathcal{I}'\subset\mathcal{I}$ containing the root $\emptyset$, and edges $\mathcal{E}'$ induced by $\mathcal{E}$. A knot $I\in\mathcal{I}'$ is called a *leaf* of $\mathcal{T}'$ if it has no children. The set of all leaves is denoted as $\mathcal{L}(\mathcal{T}')$. A leaf $I$ is called *contractive*, if the corresponding product $A_I$ is contractive. A subtree is called *finite*, if the set of knots is finite, and it is called *strong*, if every knot is either a leaf or has two children. The following lemma characterizes contractivity in terms of trees. It appears already in [1,4] (and perhaps earlier), but we include a proof for the convenience of the reader.

**Lemma 2.1.** *The set $\mathcal{A}$ is contractive if and only if there exists a finite strong subtree $\mathcal{T}'$ of $\mathcal{T}$ with contractive leaves.*

**Proof.** Suppose $\mathcal{A}$ is contractive, i.e., $\varrho(\mathcal{A})<1$. Then there exists an $n\in\mathbb{N}$, such that $\|A_I\|<1$ for all $I$ with $|I|\geqslant n$. Hence, all leaves of the subtree $\mathcal{T}'=(\mathcal{I}',\mathcal{E}')$ defined by $\mathcal{I}':=\{I\in\mathcal{I}:|I|\leqslant n\}$ are contractive.

Conversely, let $\mathcal{T}'=(\mathcal{I}',\mathcal{E}')$ be a strong finite subtree of $\mathcal{T}$ with contractive leaves, i.e., $m:=\max\{\|A_I\|: I\in\mathcal{L}(\mathcal{T}')\}<1$. Then the numbers $\ell:=\max\{|I|: I\in\mathcal{I}'\}$ and $M:=\max\{\|A_I\|:|I|<\ell\}$ are finite. Consider a knot $I\in\mathcal{I}$ with length $n:=|I|\geqslant\ell$. Because the subtree is assumed to be strong, there exists a leave $J_1\in\mathcal{L}(\mathcal{T}')$ with $J_1\leqslant I$, i.e., $I=(J_1,I_1)$. Repeating this argument for $I_1$, and so on, we find a partitioning $I=(J_1,J_2,\ldots,J_k,I_k)$, where the last vector $I_k$ admits no further split, i.e., $|I_k|<\ell$, and the remaining vectors $J_1,\ldots,J_k$ are contractive leaves of $\mathcal{T}'$. The number of these factors is at least $k\geqslant n/\ell$. Corresponding to the partitioning of $I$, we find the factorization

$$A_I = A_{I_k}A_{J_k}\cdots A_{J_1},$$

which yields the estimate $\|A_I\|\leqslant Mm^k\leqslant Mm^{n/\ell}$. Hence,

$$\|A_I\|^{1/n}\leqslant M^{1/n}m^{1/\ell}<1$$

for $n>-\ell\ln M/\ln m$, and therefore $\varrho(\mathcal{A})<1$. $\square$

In order to establish contractivity of $\mathcal{A}$, the criterion (3) suggests to traverse the tree $\mathcal{T}$ with a breadth-first search, level by level. By contrast, Lemma 2.1 shows that a depth-first search is equally possible. Typically, the latter approach is *much* more efficient.

For example, when checking the matrices $\mathcal{A}$ of the four-point scheme with parameter $\omega=0.19$ for contractivity, breadth-first search requires to consider several billions of products, while depth-first search manages with less than 300. Fig. 3 shows the left hand side of the depth-first search-tree. For symmetry reasons, the right hand side is just a mirrored version of it. The numbers attached to the knots are upper bounds of the norms of the corresponding matrix product. For example, $\|A_{(0,0,1,0)}\|\leqslant 1.01$. Contractive leaves are marked by black dots, while all other knots are marked by circles. For levels $n\geqslant 6$, we observe a repeating pattern: At each level, there exist exactly four contractive and four non-contractive knots, all characterized by an alternating prefix $(0,1,0,1,\ldots)$ and a suffix with length $\leqslant 4$. When reaching level $n=38$, the pattern is terminated by a row with only contractive leaves. Due to space limitations, this part of the tree is not shown in the figure.

While cutting branches at contractive knots can reduce the number of products to be checked considerable, the procedure still suffers from the fact that this number is hard to bound a priory. In particular, it can be excessively large if $\varrho(\mathcal{A})$ is close to 1. Instead of investigating finite, but potentially very large subtrees there are situations where it is more convenient to consider infinite subtrees with a special structure.

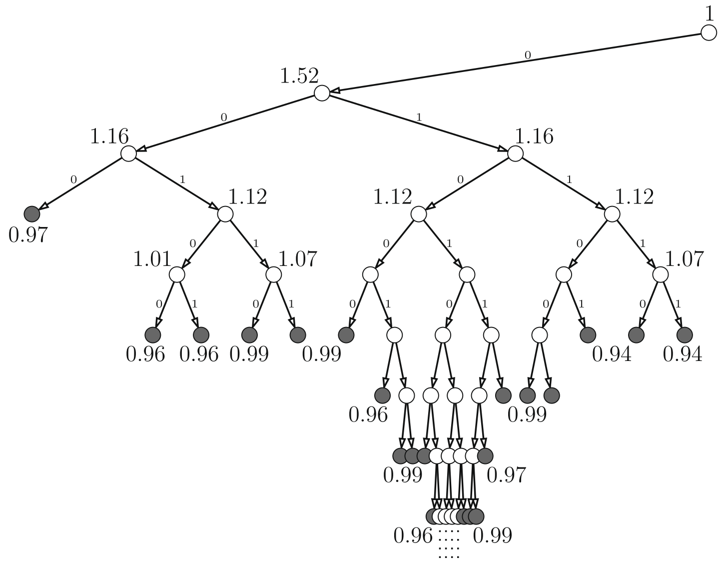


**Fig. 3.** Left hand side of strong subtree with contractive leaves for $\omega = 0.19$.

**Definition 2.2.** Denote by

$$J^k := (J, \ldots, J), \quad |J^k| = k|J|,$$

the $k$-fold repetition of the index vector $J$. For a finite set $\mathcal{J} = \{J_1, \ldots, J_m\} \subset \mathcal{I}$ of *generators* $J_\mu$ and a level $\ell \geqslant \max_{J \in \mathcal{J}} |J|$ we define the $(\mathcal{J}, \ell)$*-subtree* $\mathcal{T}' = (\mathcal{I}', \mathcal{E}')$ as the subtree of $\mathcal{T}$ with knots

$$\mathcal{I}' = \{(J^k, I) \in \mathcal{I} : J \in \mathcal{J}, k \in \mathbb{N}_0, |I| \leqslant \ell\}.$$

The condition $\ell \geqslant \max_{J \in \mathcal{J}} |J|$ ensures that $\mathcal{T}'$ is a *strong* subtree of $\mathcal{T}$. The representation of $\mathcal{I}'$ chosen above is simple, but has the disadvantage that several elements are specified more than once. For instance, we have $(J^{k+1}, \emptyset) = (J^k, J)$. To reduce redundancy, we exclude vectors $I$ with prefix $J$ and write

$$\mathcal{I}' = \{(J^k, I) \in \mathcal{I} : J \in \mathcal{J}, k \in \mathbb{N}_0, |I| \leqslant \ell, J \nleqslant I\}.$$

The leaves of $\mathcal{T}'$ satisfy

$$\mathcal{L}(\mathcal{T}') \subset \{(J^k, I) \in \mathcal{I} : J \in \mathcal{J}s, k \in \mathbb{N}_0, |I| = \ell, J \nleqslant I\}. \tag{4}$$

The following lemma provides necessary and sufficient conditions in terms of $(\mathcal{J}, \ell)$-subtrees.

**Lemma 2.3.** *Let $\mathcal{J} = \{J_1, \ldots, J_m\}$ be a set of generators and $\ell \in \mathbb{N}_0$. The set $\mathcal{A}$ is*

- *not contractive, if*

  $$\varrho_{\mathcal{J}} := \max_{J \in \mathcal{J}} \varrho(A_J) \geqslant 1,$$

  *where $\varrho(A_J)$ denotes the standard spectral radius of $A_J$.*
- *contractive, if $\varrho_{\mathcal{J}} < 1$ and the $(\mathcal{J}, \ell)$-subtree of $\mathcal{T}$ has contractive leaves.*

**Proof.** Suppose $\varrho(A_J) \geqslant 1$ for some generator $J \in \mathcal{J}$. Then

$$\|A_{J^k}\| = \|(A_J)^k\| \geqslant \varrho((A_J)^k) = \varrho(A_J)^k \geqslant 1$$

for all $k \in \mathbb{N}$. Hence, $\limsup_{n\to\infty} \sup_{|I|=n} \|A_I\|^{1/n} \geqslant 1$, and $\mathcal{A}$ is not contractive.

Conversely, suppose $\varrho_{\mathcal{J}} < 1$. Because $\lim_{k\to\infty}(A_J)^k = 0$ for all $J \in \mathcal{J}$, there exists $k_0 \in \mathbb{N}$ such that

$$\|(A_J)^{k_0}\| = \|A_{J^{k_0}}\| < 1$$

for all $J \in \mathcal{J}$. The leaves of the finite strong subtree

$$\mathcal{I}' := \{(J^k, I) : J \in \mathcal{J}, k < k_0, |I| \leqslant \ell, J \nleq I\} \cup \{J^{k_0} : J \in \mathcal{J}\}$$

are all contractive because they are either leaves of $\mathcal{I}'$ or belong to the second set on the right hand side. Hence, by Lemma 2.1, $\mathcal{A}$ is contractive. $\square$

At first sight, the sufficient condition of the lemma is not easier to verify than those given above, but its relevance can be explained as follows:

We consider the generators $J \in \mathcal{J}$ in turn and try to establish contractivity of the related leaves $(J^k, I)$ separately. Let the eigenvalues $\lambda_j$ of $A_J$ be ordered by modulus, i.e., $|\lambda_1| \geqslant \cdots \geqslant |\lambda_d|$. The spectral radius $\varrho(A_J) = |\lambda_1|$ determines the behavior of the powers of $A_J$, i.e., $\|A_J^k\| \sim |\lambda_1|^k$ as $k \to \infty$. If $|\lambda_1| \geqslant 1$, then contractivity is disproved. Otherwise, if $|\lambda_1| < 1$, we have two options: First, we observe that there exists a constant $k_0 \in \mathbb{N}$ with $\|A_{(J^k,I)}\| < 1$ for all $k \geqslant k_0$ and $I$ with $|I| \leqslant \ell$. Hence, number of leaves with $k < k_0$ that remain to be checked is finite. However, this procedure may be ineffective if $|\lambda_1|$ is only slightly smaller than 1 since then $k_0$ becomes very large. Second, we may proceed as follows:

Assuming, for simplicity, that the dominant eigenvalue $\lambda_1$ is simple and positive, we define the rescaled matrices

$$\widetilde{A}_i := \alpha_J A_i, \quad \alpha_J := \lambda_1^{-1/|J|}, \quad i \in \{0, 1\} \tag{5}$$

to obtain $\varrho(\widetilde{A}_J) = 1$. Since $\widetilde{A}_J$ has the simple eigenvalue 1, also the powers of $\widetilde{A}_J$ are converging. But now, with $u_J = u_J\widetilde{A}_J$ and $v_J = \widetilde{A}_J v_J$ denoting the left and right eigenvector of $\widetilde{A}_J$ to the eigenvalue 1, respectively, the limit

$$\widetilde{A}_J^\infty := \lim_{k\to\infty} \widetilde{A}_J^k = \frac{v_J u_J}{u_J v_J}$$

is different from 0. Here, the rate of convergence is determined by the second largest eigenvalue $\tilde{\lambda}_2 := \lambda_2/\lambda_1$ of $\widetilde{A}_J$, which may be much smaller than $\lambda_1$,

$$\|\widetilde{A}_J^k - \widetilde{A}_J^\infty\| \sim |\tilde{\lambda}_2|^k.$$

Let $\nu \in \mathbb{N}_0$ be chosen such that

$$r := \|\widetilde{A}_J^\nu - \widetilde{A}_J^\infty\| < 1 \tag{6}$$

and define

$$R := \max_{k<\nu} \|\widetilde{A}_J^k - \widetilde{A}_J^\infty\|. \tag{7}$$

Then, for given $k$, we write $k = p\nu + q$ with $q < \nu$, and obtain

$$\|\widetilde{A}_J^k - \widetilde{A}_J^\infty\| = \|(\widetilde{A}_J - \widetilde{A}_J^\infty)^k\| \leqslant \|(\widetilde{A}_J - \widetilde{A}_J^\infty)^\nu\|^p \cdot \|(\widetilde{A}_J - \widetilde{A}_J^\infty)^q\| \leqslant Rr^p.$$

Let us define the constants

$$C_1 := \max\{\|A_I\widetilde{A}_J^\infty\| : |I| = \ell, J \nleq I\}, \tag{8}$$

$$C_2 := \max\{\|A_I\| : |I| = \ell, J \nleq I\}. \tag{9}$$

According to (4), the leaves of the $(\mathcal{J}, \ell)$-subtree $\mathcal{T}$ have the form $(J^k, I)$, where the index vector $I$ has length $|I| = \ell$, and satisfies $J \nleq I$. Now,

$$\|A_{(J^k,I)}\| \leqslant \|A_I\widetilde{A}_J^k\| \leqslant \|A_I\widetilde{A}_J^\infty\| + \|A_I(\widetilde{A}_J^k - \widetilde{A}_J^\infty)\| \leqslant C_1 + C_2Rr^p.$$

If $C_1 < 1$, we can find $p_0 \in \mathbb{N}$ such that $C_1 + C_2Rr^p < 1$ for all $p \geqslant p_0$. Hence, with $k_0 := p_0\nu$, we obtain the estimate

$$\|A_{(J^k,I)}\| < 1, \quad |I| = \ell,\ k \geqslant k_0.$$

Eventually, it remains to check

$$C_3 := \max\{\|A_{(J^k,I)}\| : k < k_0, |I| = \ell, J \nleqslant I\} < 1 \tag{10}$$

to establish contractivity of all leaves in the branch of the subtree $\mathcal{T}$ generated by $J$. Let us briefly discuss some aspects of this approach.

- The value of $\nu$ has to be sufficiently large to ensure (6), but otherwise, it can be chosen arbitrarily.
- Given $\nu$, the constants $r, R, C_1, C_2, C_3$ can be determined by computing finitely many matrix products. Further, in the application we have in mind, the total number of products to be considered is bounded *independently* of the parameter $\omega$, despite the fact that the size of the depth-first search-tree is becoming arbitrarily large.
- If the matrices depend on a parameter $\omega$ varying in some interval, one can sample the quantities in question on a sufficiently dense subset to obtain a reliable indication of contractivity on the whole interval. A *rigorous* verification might be tricky, but can be accomplished in special cases. We will come back to this point in the context of the four-point scheme at the end of the paper.
- The concept of $(\mathcal{J}, \ell)$-subtrees is kept as simple as possible for the sake of clarity. It is possible to devise even smaller subtrees with similar properties at the cost of more involved definitions.

## 3. $C^1$-Continuity

Now, we are going to verify our main result:

**Theorem 3.1.** *The set $\Omega$ of $C^1$-parameters of the four-point-scheme is $\Omega = (0, \omega^*)$, where*

$$\omega^* := \frac{1}{12}(27 + 3\sqrt{105})^{1/3} - \frac{1}{2}(27 + 3\sqrt{105})^{-1/3} \approx 0.19273$$

*is the unique real solution of the equation $32\omega^3 + 4\omega - 1 = 0$.*

**Proof.** With $A_0^\omega, A_1^\omega$ according to (2) and $\mathcal{A}^\omega := \{A_0^\omega, A_1^\omega\}$, we have to show that

$$0 < \omega < \omega^* \Leftrightarrow \varrho(\mathcal{A}^\omega) < 1.$$

From now on, we omit the superscript $\omega$ to simplify notation. Dyn et al. [7] show $(0, 1/8] \subset \Omega \subset (0, 1/4)$, and later on [8] $(0, (\sqrt{5} - 1)/8] \subset \Omega \subset (0, 1/4)$. Villemoes [11] improves the upper bound to $\Omega \subset (0, \omega^*)$ by using the estimate $\varrho(\mathcal{A})^2 \geqslant \varrho(A_{(0,1)}) = \varrho(A_{(1,0)}) = (\omega^*)^2$.

To improve the lower bound, we consider all products $A_I$ with $|I| = 4$. The entries are quartic polynomials in $\omega$, which can be computed using a computer algebra system. Skipping the details, $\|A_I\| < 1$ for all products at level 4 if

$$\max\{\|A_{(0,0,0,0)}\|, \|A_{(0,1,0,1)}\|\} < 1.$$

For $\omega > 1/8$, this condition reads $8\omega^2 + 160\omega^3 - 64\omega^4 < 1$, what is satisfied if $0 < \omega \leqslant 0.17$. On the left hand side, Fig. 4 illustrates the result by showing all $\|A_I\|, |I| = 4$, as a function of $\omega$. The thicker lines correspond to $I = (0, 0, 0, 0)$ and $I = (0, 1, 0, 1)$, respectively. Hence, it remains to show that

$$\omega_* < \omega < \omega^* \Rightarrow \varrho(\mathcal{A}^\omega) < 1, \quad \omega_* := 0.17.$$

With

$$A := \begin{bmatrix} 0 & 0 & 4\omega & 4\omega \\ 0 & -2\omega & 1 - 4\omega & -2\omega \\ 0 & 4\omega & 4\omega & 0 \\ -2\omega & 1 - 4\omega & -2\omega & 0 \end{bmatrix}, \quad P := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

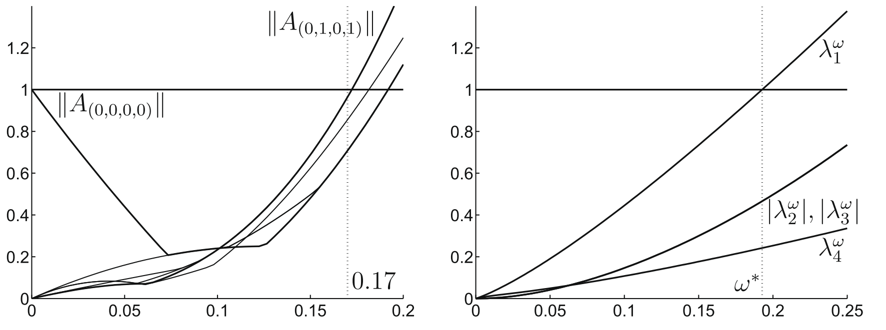


**Fig. 4.** Norm of the products $A_I$, $|I| = 4$, for $\omega \in [0, 0.2]$ (*left*) and eigenvalues of $A$ for $\omega \in [0, 1/4]$ (*right*).

we obtain $A_0 = AP$ and $A_1 = PA$. Because multiplication by $P$ just amounts to a permutation of rows or columns, the spectra of any $(4 \times 4)$-matrix $B$ and $PBP$ coincide. In particular,

$$\varrho(B) = \varrho(PBP). \tag{11}$$

This symmetry relation will help to simplify the subsequent arguments.

Now, we consider the $(\mathcal{J}, 6)$-subtree $\mathcal{T}'$ given by the generators

$$\mathcal{J} := \{(0, 1), (1, 0)\},$$

as suggested in Fig. 3. The level $\ell = 6$ is chosen slightly larger than the optimal value $\ell = 4$ to retain an extra margin for our estimates. In what follows, we assume $\omega > 0$ without further notice. With

$$A_{(0,1)} = PA^2P \quad \text{and} \quad A_{(1,0)} = A^2,$$

we find, using (11), $\varrho_{\mathcal{J}} = \varrho(A^2) = \varrho(A)^2$ for the constant defined in Lemma 2.3. The characteristic polynomial of $A$ is

$$p(\lambda) := \lambda^4 - 2\lambda^3\omega + (8\omega^2 - 2\omega)\lambda^2 - 8\omega^2\lambda - 32\omega^3, \tag{12}$$

which has two real and a pair of complex roots. As illustrated by Fig. 4 (*right*), the moduli of the complex roots and one of the real roots are $< 0.6$ for $\omega < 0.2$, Hence, the spectral radius of $A$ is determined by the larger real root of $p$, and $\varrho(A) = 1$ if and only if

$$p(1) = 1 - 4\omega - 32\omega^3 = 0.$$

The unique real solution of this equation is $\omega^*$, as given in the theorem, so that $\varrho(A) < 1$ if and only if $\omega < \omega^*$. By the first part of Lemma 2.3, this shows that contractivity is impossible for $\omega \geqslant \omega^*$, confirming the result of Villemoes.

Now we establish contractivity for $\omega \in (\omega_*, \omega^*)$, following the directions given in the sequel of Lemma 2.3. We consider the generator

$$J = (0, 1)$$

for the remainder of the proof. For symmetry reasons, the results for the other one are completely the same. The eigenvalues of $A_J$ are squares of the eigenvalues of $A$, showing that the largest eigenvalue $\lambda_1$ of $A_J$ is real and simple. Hence, we may proceed with determining the constants $r, R, C_1, C_2, C_3$ defined in Eqs. (6)–(10). Of course, these constants depend on $\omega$, but we are able to specify *uniform* upper bounds, which are valid for all $\omega \in (\omega_*, \omega^*)$. In Fig. 5, the norms of all matrices in question are plotted as functions of $\omega$ for $\nu = 2$. The thick horizontal lines indicate the following uniform upper bounds:

$$r = \|\widetilde{A}_J^2 - \widetilde{A}_J^\infty\| < 0.44,$$
$$R = \max\{\|\mathrm{Id} - \widetilde{A}_J^\infty\|, \|\widetilde{A}_J - \widetilde{A}_J^\infty\|\} < 1.58,$$
$$C_1 = \max\{\|A_I\widetilde{A}_J^\infty\| : |I| = 6, J \not\leqslant I\} < 0.92,$$
$$C_2 = \max\{\|A_I\| : |I| = 6, J \not\leqslant I\} < 1.5.$$

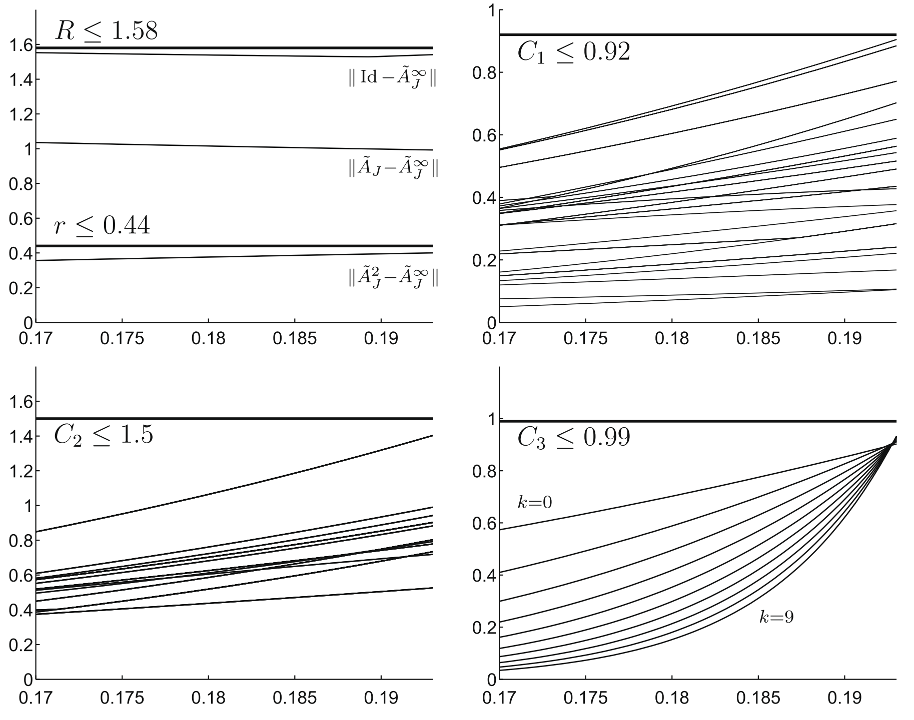


**Fig. 5.** Bounds on the constants depending on $\omega \in (\omega_*, \omega^*)$.

Some comments concerning the machinery required for a *rigorous* verification of the given bounds is postponed to the end of this section.

With $k = 2p + q$, $q < 2$, the matrix products corresponding to leaves $(J^k, I)$ of the subtree $\mathcal{T}'$ satisfy

$$\|A_{(J^k,I)}\| \leqslant C_1 + C_2 R r^p \leqslant 0.92 + 1.5 \cdot 1.58 \cdot 0.44^p \leqslant 0.99$$

for $p \geqslant p_0 := 5$. Finally, we obtain the uniform bound

$$C_3 = \max\{\|A_{(J^k,I)}\| : k < 10, |I| = 6, J \nleqslant I\} \leqslant 0.99$$

on the remaining 480 leaves with $k < 10$. Due to the quite large number of products, Fig. 5 (*bottom, right*) shows only the maximal norms for the exponents $k = 0, 1, \ldots, 9$.

As mentioned above, the second generator $J = (1, 0)$ yields exactly the same results so that $\mathcal{A}$ is contractive by Lemma 2.3 for all $\omega \in (\omega_*, \omega^*)$. □

Of course, Fig. 5 must not be confused with a rigorous proof for the given bounds on the constants. The plots were generated using a quite large number of $\omega$-values, but still, there might be nasty surprises lurking between the samples. Below, we describe the ideas behind some computer algebra programs which we have implemented to actually verify the bounds.

The constants $C_2$ and $C_3$ are treated as follows: The elements of the related matrices are polynomials in $\omega$ so that estimates of the form

$$\sum_{i=1}^{4} |p_i(\omega)| < C_{2,3}, \quad \omega \in (\omega_*, \omega^*)$$

have to be verified for always 4 polynomials $p_i$ forming the rows of these matrices. Equivalently, we can show that

$$\sum_{i=1}^{4} s_i p_i(\omega) - C_{2,3} \neq 0,\ \ \omega \in (\omega_*, \omega^*) \quad \text{and} \quad \sum_{i=1}^{4} s_i p_i(\omega_*) - C_{2,3} < 0$$

for all 16 combinations of signs $s_i \in \{-1, 1\}$. The left hand side of the first condition is a single polynomial in $\omega$, and the absence of roots in the given interval can be established using Sturm sequences. To speed up calculations, it is convenient to enlarge the interval slightly, say $\omega \in [17/100, 193/1000]$, to obtain rational bounds. The verification of the second condition is straightforward.

For the other constants, the demanding part concerns the determination of the limit $\widetilde{A}_J^\infty$. Rescaling $A$ by

$$\widetilde{A} := \alpha A, \quad \alpha := \varrho(A)^{-1}$$

yields $\widetilde{A}_J^k = P\widetilde{A}^{2k}P$ for all $k \in \mathbb{N}_0$ and

$$\widetilde{A}_J^\infty = P\widetilde{A}^\infty P, \quad \widetilde{A}^\infty := \lim_{k\to\infty} \widetilde{A}^k.$$

Instead of computing $\widetilde{A}_J^\infty$ directly, it is more convenient to determine $\widetilde{A}^\infty$ and to use the above relation because $\widetilde{A}_J$ is quadratic and $\widetilde{A}$ is only linear in $\omega$.

With $p$ as given in (12), the characteristic polynomial of $\widetilde{A}$ is $\tilde{p}(\lambda) = \alpha^4 p(\lambda/\alpha)$. Hence, the equation

$$\tilde{p}(1) = -32\alpha^4\omega^3 - 8\alpha^3\omega^2 + \alpha^2(8\omega^2 - 2\omega) - 2\alpha\omega + 1 = 0$$

characterizes the scaling factor $\alpha$ as a function of $\omega$. Unfortunately, the resulting explicit expression is far to complicated to be useful for further processing. Alternatively, we substitute $\alpha := \beta/(2\omega)$ and solve for $\omega$ to obtain the equivalent, but much simpler condition

$$\omega = \frac{\beta^2(4\beta^2 + 2\beta + 1)}{2(2\beta^2 - \beta + 1)}. \tag{13}$$

Using this equation, we now regard $\omega = \omega(\beta)$ as a function of the new parameter $\beta$. It is easily verified by inspection that the values of $\omega$ cover the interval $(\omega_*, \omega^*)$ if $\beta$ varies in the interval $[0.367, 0.386]$. Now, the scaling factor

$$\alpha = \frac{\beta}{2\omega} = \frac{2\beta^2 - \beta + 1}{\beta(4\beta^2 + 2\beta + 1)}$$

is a rational function of $\beta$, so that the elements of

$$\widetilde{A} = \begin{bmatrix} 0 & 0 & 2\beta & 2\beta \\ 0 & -\beta & \alpha - 2\beta & -\beta \\ 0 & 2\beta & 2\beta & 0 \\ -\beta & \alpha - 2\beta & -\beta & 0 \end{bmatrix}$$

are rational in $\beta$, too. The left eigenvector $u = u\widetilde{A}$ and the right eigenvector $v = \widetilde{A}v$ of $\widetilde{A}$ to the eigenvalue 1 are given by

$$u^{\mathrm{t}} = \begin{bmatrix} 2\beta^3 \\ 2\beta(2\beta^2 + 1) \\ 2\beta^3 + \alpha\beta + \beta + 1 \\ -2\beta^2 \end{bmatrix}, \quad v = \begin{bmatrix} 2\beta(2\alpha\beta + \beta - 1) \\ \beta(1 - 2\beta) \\ 2\beta^2 \\ 2\alpha\beta - 2\beta^2 + \beta - 1 \end{bmatrix}$$

and define the limit matrix via

$$\widetilde{A}^\infty = \frac{vu}{uv}.$$

The least common denominator

$$q_1(\beta) := \beta^2(1 + 2\beta + 4\beta^2)(2 + 5\beta + 12\beta^2 - 8\beta^3 + 16\beta^4)$$

of this matrix is positive on $(\omega_*, \omega^*)$. Let us consider, for example, the constant $C_1$; similar considerations apply to $r$ and $R$. Using (13), the product $A_I, |I| = 6$, is written in dependence of $\beta$. Again, the least common denominator

$$q_2(\beta) := (1 - \beta + 2\beta^2)^6$$

is positive. Now, we have to verify estimates of the form

$$\sum_{i=1}^{4} \frac{|p_i(\omega)|}{q_1(\omega)q_2(\omega)} < C_1, \quad \omega \in (\omega_*, \omega^*).$$

Because $q_1 q_2 > 0$, it is equivalent to show

$$\sum_{i=1}^{4} s_i p_i(\omega) - C_1 q_1(\omega) q_2(\omega) \neq 0, \quad \omega \in (\omega_*, \omega^*)$$

and

$$\sum_{i=1}^{4} s_i p_i(\omega_*) - C_1 q_1(\omega_*) q_2(\omega_*) < 0$$

for all 16 combinations of signs $s_i \in \{-1, 1\}$. As above, these inequalities involve only polynomials and can be proven using Sturm sequences. The total runtime for all 48 products using Maple 11 on a standard PC is less than 2 min.

## 4. Conclusion

The determination of the set $\Omega = (0, \omega^*)$ of $C^1$-tension parameters for the four-point scheme solves an open problem of long standing. The methods employed here cannot claim universal applicability, but they might be useful in other cases as well. Future research in this direction will address the analysis of other subdivision schemes, and the problem of determining sharp Hölder exponents of the limit curves of the four-point scheme in dependence of $\omega$.

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