On Curve Interpolation in $\mathbb{R}^d$

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Abstract. In this paper the interpolation by G^2 continuous spline curves of degree n in $\mathbb{R}^d$ is studied. There are r interior and two boundary data points interpolated on each segment of the spline curve. The general form of the spline curve, as well as the defining system of nonlinear equations are derived. The asymptotic existence of the solution, and the approximation order are studied for the polynomial case only. It is shown that the optimal approximation order is achieved, and asymptotic existence is established provided the relation $r = n - 2$ is satisfied. These conclusions hold independently of d. It is also pointed out that the underlying analysis could not be carried over to the case $r = n - 1$.

§1. Introduction

The interpolation problem considered is the following. Let the points

$$\boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_N \in \mathbb{R}^d, \quad \boldsymbol{T}_j \neq \boldsymbol{T}_{j+1}, \text{ all } j, \ d \geq 2, \tag{1}$$

and the tangent directions

$$\boldsymbol{d}_0, \boldsymbol{d}_N \tag{2}$$

at the boundary points be given. Find a G^2 continuous spline curve $\boldsymbol{B}_n$ of degree n which interpolates the prescribed data.

The problem appeared first as a particular limit case in [2], and was further generalized in several papers, among them in [3–5, 6, 9–10]. A general approach to the approximation order achieved can be found in [8].

Here, the general setup is tackled. The interpolating spline curve in the Lagrange form is established and the defining system of nonlinear equations is derived in general. However, the asymptotic existence of the solution (i.e. the existence of the solution when given points are sampled densely enough) and the approximation order turned out too comprehensive to be studied here in a general framework. The positive conclusions for the single segment case when $r = n - 2$ are established. It is possible to extend these results to the m-segment spline curve, but the proofs are not short, and will appear

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elsewhere. On the contrary, as one could guess from [8], the case $r = n - 1$ is not encouraging.

Why would one use the G^2-continuous splines as interpolating curves? Quite clearly, the derivative continuity at the breakpoints becomes in this way independent of the local parametrisation. Also, these curves could be seen as a generalization of the odd order spline function interpolation at knots, applied so successfully in many cases. The order of G-continuity 2 is pinned down by the human eye, sometimes quite important in CAGD: it can detect the continuity, the continuity of the tangent direction and the curvature, but hardly higher order geometric quantities.

Throughout the paper bold faced letters will stand for vectors, and ordinary ones for scalars. The dot product on $\mathbb{R}^d$ will be denoted by $\cdot$ and its implied norm by $||\cdot||$. Derivatives with respect to the global (or local) parameter will be denoted by $\dot{}$ (or $d/d\zeta$), and those with respect to the natural parameter by $'$.

Now let $\boldsymbol{B}_n$ be a continuous spline curve of degree n with m segments

$$\boldsymbol{B} := \boldsymbol{B}_n : [\zeta_0, \zeta_m] \to \mathbb{R}^d$$

with breakpoints

$$\zeta_0 < \zeta_1 < \ldots < \zeta_m,$$

given piecewise as

$$\boldsymbol{B}(\zeta) = \boldsymbol{B}^\ell\left(\frac{\zeta - \zeta_{\ell-1}}{\Delta\zeta_{\ell-1}}\right), \quad \zeta \in [\zeta_{\ell-1}, \zeta_\ell],$$

i.e., locally parametrized on $[0,1]$. Suppose $\boldsymbol{B}$ interpolates the data (1), and (2). If r interior and two boundary points are to be met on each segment, then $N = m(r+1)$. Further, on the ℓ-th segment the interpolation conditions read

$$\boldsymbol{B}^\ell(t_{\ell,j}) = \boldsymbol{T}_{\ell,j} := \boldsymbol{T}_{(\ell-1)(r+1)+j}, \quad j = 0, 1, \ldots, r+1, \quad \ell = 1, 2, \ldots, m, \tag{3}$$

where

$$0 =: t_{\ell,0} < t_{\ell,1} < \cdots < t_{\ell,r+1} := 1,$$

and $(t_{\ell,j})_{j=1}^r$ are the unknown parameters to be determined. Let $\boldsymbol{x} \wedge \boldsymbol{y} \mapsto (x_i y_j - x_j y_i)_{i<j}$ denote the 2-wedge product. The geometric continuity of $\boldsymbol{B}$ requires the tangent direction

$$\frac{1}{||\dot{\boldsymbol{B}}||}\dot{\boldsymbol{B}} \tag{4}$$

as well as the curvature

$$\frac{1}{||\dot{\boldsymbol{B}}||^3}\dot{\boldsymbol{B}} \wedge \ddot{\boldsymbol{B}} \tag{5}$$

to be continuous at the breakpoints. Additionally, at the boundary points the tangent directions $\boldsymbol{d}_0$ and $\boldsymbol{d}_N$ have to be interpolated too, i.e.,

$$\boldsymbol{d}_0 \wedge \dot{\boldsymbol{B}}(\zeta_0) = \dot{\boldsymbol{B}}(\zeta_m) \wedge \boldsymbol{d}_N = \boldsymbol{0}. \tag{6}$$

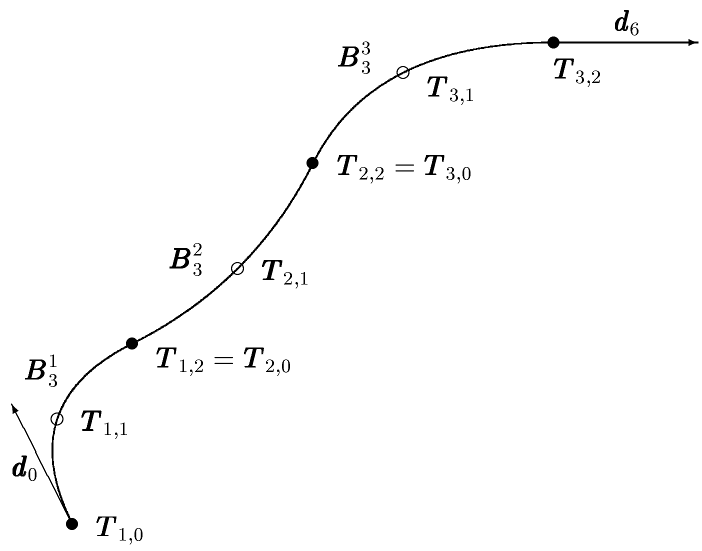

Fig. 1. An interpolating spline curve with three segments .

Fig. 1 gives an example of such an interpolating spline curve for $r = 1$, $n = 3$, and $d = 3$. A brief look at the conditions (3)–(6) reveals that the number of independent equations would be equal to the number of independent unknowns if

$$d\,n - (d-1)\,r = 3\,d - 2. \tag{7}$$

As already observed in [5], for fixed d this Diophantine equation always has an infinite number of nonnegative solutions. The following lemma gives its general solution.

Lemma 1. *The possible choices of pairs r and n that satisfy (7) for fixed d are given by*

$$r = d - 2 + dk, \quad n = d + (d-1)k, \quad k = 0, 1, \ldots. \tag{8}$$

Proof: The relation (7) can be rewritten as

$$d(n-d) - (d-1)(r-d+2) = 0.$$

Since $d \geq 2$, the numbers d and $d-1$ are relatively prime. So d must divide $r - d + 2$, and $d - 1$ must divide $n - d$, i.e.,

$$\frac{r-d+2}{d} = \frac{n-d}{d-1} = k$$

for an integer k. But $r = d - 2 + dk \geq 0$ implies $k \geq \frac{2}{d} - 1 > -1$, and the conclusion follows. $\square$

§2. The Defining Equations

Several approaches were used to simplify the conditions (3)–(6) for particular choices of d, n, and r. Here we show that this can be done in general, which will provide an opportunity to unify the computer programs. Let us consider a single segment first. In this case, the data to be interpolated are the points $\boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_{r+1}$, $\boldsymbol{T}_j \neq \boldsymbol{T}_{j+1}$, as well as tangent directions $\boldsymbol{d}_0$, $\boldsymbol{d}_{r+1}$ at the boundary points. Suppose r and n are given by (8). Consider the case $n = r+2$ first, i.e., $k = 0$. The interpolating polynomial curve can be written explicitly in Lagrange form as

$$\boldsymbol{B} := \boldsymbol{b}\,\omega + \sum_{j=0}^{r+1} \boldsymbol{T}_j \mathcal{L}_j$$

with

$$\omega(t) := \prod_{j=0}^{r+1} (t - t_j), \quad \mathcal{L}_j(t) := \frac{\omega(t)}{(t - t_j)\omega'(t_j)}, \tag{9}$$

$t_j := t_{1,j}$, and the values $(t_j)_{j=1}^{r}$, to be determined. Here $\boldsymbol{b} \in \mathbb{R}^d$ denotes the unknown leading coefficient vector. If $k \geq 1$, one has

$$r + 2 = d(k+1) > d(k+1) - k = n,$$

and $\boldsymbol{B}$ is of degree at most $r+1$, i.e.,

$$\boldsymbol{B} = \sum_{j=0}^{r+1} \boldsymbol{T}_j \mathcal{L}_j.$$

In particular, this imposes additional conditions

$$\text{degree} \sum_{j=0}^{r+1} \boldsymbol{T}_j \mathcal{L}_j \leq n \tag{10}$$

for $k > 1$. An easy way to meet the tangent direction conditions (6) is to introduce two additional (strictly positive) real unknowns, α_0 and α_{r+1}, and require

$$\dot{\boldsymbol{B}}(t_0) = \alpha_0 \boldsymbol{d}_0, \quad \dot{\boldsymbol{B}}(t_{r+1}) = \alpha_{r+1} \boldsymbol{d}_{r+1}. \tag{11}$$

Let

$$\tau_{-1} = \tau_0 := t_0, \quad \tau_j := t_j, \; j = 1, 2, \ldots, r, \quad \tau_{r+2} = \tau_{r+1} := t_{r+1}. \tag{12}$$

Since $\boldsymbol{B}$ is a polynomial of degree $\leq n$, the divided difference, based upon

$$n + 2 = r + 4 - k$$

points maps it to zero. So the conditions (11) and (10) can be written in a compact form as

$$[\tau_{j-1}, \tau_j, \ldots, \tau_{j+r+2-k}]\boldsymbol{B} = 0, \; j = 0, 1, \ldots, k, \tag{13}$$

which is a system of $d(k+1)$ nonlinear equations for $r+2 = d(k+1)$ scalar unknowns

$$\alpha_0, t_1, t_2, \ldots, t_r, \alpha_{r+1}. \tag{14}$$

In the case $n = r+2$, one has to determine additionally the coefficient vector $\boldsymbol{b}$, for example as

$$\boldsymbol{b} = [t_0, t_0, t_1, \ldots, t_r, t_{r+1}]\boldsymbol{B} = [t_0, t_1, \ldots, t_r, t_{r+1}, t_{r+1}]\boldsymbol{B}. \tag{15}$$

Now, for an m-segment spline curve, the directions $\boldsymbol{d}_\ell$, $\ell = 1, 2, \ldots, m-1$, are unknown, as well as

$$\alpha_{\ell,0}, t_{\ell,1}, t_{\ell,2}, \ldots, t_{\ell,r}, \alpha_{\ell,r+1}, \quad \ell = 1, 2, \ldots, m. \tag{16}$$

But one can still write the interpolation conditions on the ℓ-th segment as

$$[\tau_{\ell,j-1}, \tau_{\ell,j}, \ldots, \tau_{\ell,j+r+2-k}]\boldsymbol{B}^\ell = 0, \quad j = 0, 1, \ldots, k, \tag{17}$$

where $\tau_{\ell,j}$ are defined as in (12), but this time for the composite case. In addition, the missing $(d-1)(m-1)$ equations are supplied by the continuity conditions of the curvature (5).

§3. Asymptotic Existence and Approximation Order

The system of equations based on (17) and continuity of curvature (5) is nonlinear, and one of the approaches to study it is to assume that the data (1) and (2) are based upon a smooth underlying regular parametric curve $\boldsymbol{f}: I \to \mathbb{R}^d$, parametrized by the arclength s. The local expansion of the curve $\boldsymbol{f}$, and the data $\boldsymbol{T}_{\ell,j}$ (sampled densely enough), give rise to an asymptotic analysis of the nonlinear system. The simplest way to obtain the local expansion is to use the Frenet frame as the local coordinate system, and the Frenet-Serett formulae to obtain this expansion. Let $(\boldsymbol{e}_i(s))_{i=1}^d$ denote the Frenet frame, with

$$\boldsymbol{f}' = \boldsymbol{e}_1. \tag{18}$$

The Frenet-Serret formulae read

$$\begin{aligned} \boldsymbol{e}_1'(s) &= \kappa_1(s)\boldsymbol{e}_2(s), \\ \boldsymbol{e}_i'(s) &= -\kappa_{i-1}(s)\boldsymbol{e}_{i-1}(s) + \kappa_i(s)\boldsymbol{e}_{i+1}(s), \quad i = 2, 3, \ldots, d-1, \\ \boldsymbol{e}_d'(s) &= -\kappa_{d-1}(s)\boldsymbol{e}_{d-1}(s), \end{aligned} \tag{19}$$

where κ_i are first $d-1$ principal curvatures of $\boldsymbol{f}$, expanded as

$$\kappa_i(s) = \kappa_{i0} + \frac{1}{1!}\kappa_{i1}s + \frac{1}{2!}\kappa_{i2}s^2 + \ldots. \tag{20}$$

Since $\boldsymbol{f}$ is a regular curve, $\kappa_{i0} \geq \text{const} > 0$, $i = 1, 2, \ldots, d-2$. We will additionally assume that $\kappa_{d0} \geq \text{const} > 0$. Beginning with (18), the higher derivatives of $\boldsymbol{f}$ can be computed by (19) and (20). This produces the required expansion

$$\begin{aligned}\boldsymbol{f}(s) &= \boldsymbol{f}(0) + \boldsymbol{f}^{'}(0)s + \frac{1}{2!}\boldsymbol{f}^{''}(0)s^2 + \cdots \\ &= \boldsymbol{f}(0) + (s - \frac{1}{6}\kappa_{1,0}^2 s^3 + \cdots)\boldsymbol{e}_1(0) \\ &+ (\frac{1}{2}\kappa_{1,0}s^2 + \frac{1}{6}\kappa_{1,1}s^3 + \cdots)\boldsymbol{e}_2(0) + (\frac{1}{6}\kappa_{1,0}\kappa_{2,0}s^3 + \cdots)\boldsymbol{e}_3(0) + \cdots\end{aligned} \tag{21}$$

Let us now consider the single segment case of the interpolation problem with data based on a smooth $\boldsymbol{f} : [0, h] \to \mathbb{R}^d$,

$$\boldsymbol{d}_0 = \boldsymbol{f}'(\eta_0 h), \quad \boldsymbol{T}_j = \boldsymbol{f}(\eta_j h),\ j = 0, 1, \ldots, r+1, \quad \boldsymbol{d}_{r+1} = \boldsymbol{f}'(\eta_{r+1} h),$$

with points separated independently of h, i.e.,

$$0 := \eta_0 < \eta_1 < \cdots < \eta_r < \eta_{r+1} := 1.$$

Since translation and rotation do not influence the asymptotic analysis, we may assume $\boldsymbol{f}(0) = \boldsymbol{0}$, and

$$\boldsymbol{e}_i(0) = (\delta_{i,j})_{j=1}^d, \quad i = 1, 2, \ldots, d. \tag{22}$$

Then, with the help of (21), one obtains

$$\boldsymbol{f}(\eta_j h) = \left(\frac{1}{i!}\eta_j^i h^i \prod_{q=0}^{i-1} \kappa_{q,0} + \mathcal{O}(h^{i+1})\right)_{i=1}^d, \tag{23}$$

and a similar expression for $\boldsymbol{f}'(h)$. Since the divided difference is a linear functional, we can normalize the system (13) by multiplying the data values by D^{-1}, with

$$D := \operatorname{diag}\left(\frac{1}{i!}\, h^i \prod_{q=1}^{i-1} \kappa_{q,0}\right)_{i=1}^d .$$

Let $\widetilde{\boldsymbol{f}}(s) := (s^i)_{i=1}^d$ denote the leading part of the normalized $\boldsymbol{f}$. Then

$$[t_0, t_0, t_1, \ldots, t_r, t_{r+1}, t_{r+1}]D^{-1}\boldsymbol{B} = [t_0, t_0, t_1, \ldots, t_r, t_{r+1}, t_{r+1}]\widetilde{\boldsymbol{B}} + \mathcal{O}(h), \tag{24}$$

and $\widetilde{\boldsymbol{B}}$ is a polynomial of degree $\leq n = r+2$ that satisfies the interpolation conditions

$$\begin{aligned}\widetilde{\boldsymbol{B}}^{'}(t_j) &= \widetilde{\alpha}_j \widetilde{\boldsymbol{f}}^{'}(\eta_j), \quad j = 0, r+1, \\ \widetilde{\boldsymbol{B}}(t_j) &= \widetilde{\boldsymbol{f}}(\eta_j), \quad j = 0, 1, \ldots, r, r+1,\end{aligned}$$

where

$$\widetilde{\alpha}_0 := \frac{\alpha_0}{h}, \quad \widetilde{\alpha}_{r+1} := \frac{\alpha_{r+1}}{h}.$$

Note that all the components of $\widetilde{\boldsymbol{f}}$ are polynomials of degree $\leq d = r+2$. This implies that

$$[\eta_0, \eta_0, \eta_1, \ldots, \eta_r, \eta_{r+1}, \eta_{r+1}]\widetilde{\boldsymbol{f}} = 0, \tag{25}$$

and the solution of (24) in the limit $h \to 0$ now reads as

$$\boldsymbol{t}^* := (\widetilde{\alpha}_0^*, t_1^*, t_2^*, \ldots, t_r^*, \widetilde{\alpha}_{r+1}^*) = (1, \eta_1, \eta_2, \ldots, \eta_r, 1). \tag{26}$$

To prove the existence of the solution for h small enough, it is sufficient to show that the Jacobian of the system (24) is nonsingular at the limit (26). The Jacobian will be determined with the help of the following fact: if x_j is different from all the other points x_i, and if a function g is smooth enough, one has

$$\begin{aligned}\Big(\frac{\partial}{\partial x_j}[\ldots, x_j, \ldots]\Big)g &= \frac{d}{dx_j}\Big([\ldots, x_j, \ldots]g\Big) - \frac{g'(x_j)}{\prod_{i\neq j}(x_j - x_i)} \\ &= [\ldots, x_j, x_j, \ldots]g - \frac{g'(x_j)}{\prod_{i\neq j}(x_j - x_i)}.\end{aligned} \tag{27}$$

Consider now $\widetilde{\boldsymbol{B}} = \big(\widetilde{\boldsymbol{B}} - \widetilde{\boldsymbol{f}}\big) + \widetilde{\boldsymbol{f}}$. Since $\widetilde{\boldsymbol{B}} - \widetilde{\boldsymbol{f}} = 0$ at $\boldsymbol{t}^*$, all its partial derivatives with respect to t_j vanish, and this difference contributes to the Jacobian at the limit point $\boldsymbol{t}^*$ only in the first and last column, i.e.,

$$\begin{aligned}\frac{\partial}{\partial \alpha_0}[t_0, t_0, t_1, \ldots, t_r, t_{r+1}, t_{r+1}]\big(\widetilde{\boldsymbol{B}} - \widetilde{\boldsymbol{f}}\big)\Big|_{\boldsymbol{t}^*} &= \frac{1}{(\eta_0 - \eta_{r+1})\widetilde{\omega}'(\eta_0)}\widetilde{\boldsymbol{f}}'(\eta_0), \\ \frac{\partial}{\partial \alpha_{r+1}}[t_0, t_0, t_1, \ldots, t_r, t_{r+1}, t_{r+1}]\big(\widetilde{\boldsymbol{B}} - \widetilde{\boldsymbol{f}}\big)\Big|_{\boldsymbol{t}^*} &= \frac{1}{(\eta_{r+1} - \eta_0)\widetilde{\omega}'(\eta_{r+1})}\widetilde{\boldsymbol{f}}'(\eta_{r+1}),\end{aligned} \tag{28}$$

where ω is given by (9), and

$$\widetilde{\omega} := \omega\Big|_{\boldsymbol{t}^*}.$$

The polynomial curve $\widetilde{\boldsymbol{f}}$ does not depend on $\widetilde{\alpha}_0, \widetilde{\alpha}_{r+1}$, and from (27) and (25) one obtains the columns $2, 3, \ldots, r+1$ with $j = 1, 2, \ldots, r$ as

$$\left(\frac{\partial}{\partial t_j}[t_0, t_0, t_1, \ldots, t_r, t_{r+1}, t_{r+1}]\right)\widetilde{\boldsymbol{f}}\Big|_{\boldsymbol{t}^*} = -\frac{1}{(\eta_j - \eta_0)(\eta_j - \eta_{r+1})\widetilde{\omega}'(t_j)}\widetilde{\boldsymbol{f}}'(\eta_j).$$

It is now straightforward to see that the Jacobian at $\boldsymbol{t}^*$ is the Vandermonde matrix $V(\eta_0, \eta_1, \ldots, \eta_{r+1})$, multiplied by $D_1 := \operatorname{diag}(i)_{i=1}^d$ from the left, and by

$$D_2 := \operatorname{diag}\left(-\frac{1}{\widetilde{\omega}'(\eta_0)}, \frac{1}{\eta_1(1-\eta_1)\widetilde{\omega}'(\eta_1)}, \ldots, \frac{1}{\eta_r(1-\eta_r)\widetilde{\omega}'(\eta_r)}, \frac{1}{\widetilde{\omega}'(\eta_{r+1})}\right)$$

from the right. This prepares the proof of the following theorem.

Theorem 2. *The system (13) has a unique solution for h small enough. The approximation order of the resulting interpolating polynomial curve $\boldsymbol{B}_n$ is optimal, i.e., $r+4=n+2$.*

Proof: Since the matrices $V(\eta_0, \eta_1, \ldots, \eta_{r+1})$, D_1 and D_2 are nonsingular, the Jacobian at the limit point $\boldsymbol{t}^*$ is nonsingular, too, and the existence of a unique solution for h small enough is established. Furthermore, the unknown parameters are of the form

$$\alpha_0 = \alpha_{r+1} = h + \mathcal{O}(h^2), \quad t_j = \eta_j + \mathcal{O}(h), \quad j = 1, 2, \ldots, r. \tag{29}$$

Since there are $r+2$ points, as well as two directions interpolated, the optimal approximation order is quite clearly $\leq r+4$. The proof will now follow the approach applied in [2], and extended in [5]. It is based on a reparametrisation that transforms the direction interpolation to the derivative interpolation, and gives an estimate of the parametric approximation order as defined in [7]. Recall (22), and the fact that interpolation conditions are satisfied. By [2] and [5], it is now enough to confirm that all the components of $\boldsymbol{f}$ and $\boldsymbol{B}$ can be reparametrized by the ordinate of the first component of both curves. As to $\boldsymbol{f}$, for h small enough this fact is obvious. The first component behaves by (21) as $s + \mathcal{O}(s^3)$, and the others at least as $\mathcal{O}(s^2)$. To establish the same conclusion for $\boldsymbol{B}$, it is enough to show that

$$\dot{\boldsymbol{B}} = c\, h (\delta_{1i})_{i=1}^d + \mathcal{O}(h^2), \; c \neq 0. \tag{30}$$

Further, the optimal approximation order proof depends on the additional relations

$$\boldsymbol{B}^{(q)} = \mathcal{O}(h^q), \quad q = 2, 3, \ldots, r+2. \tag{31}$$

The result required then follows from the standard error estimate of interpolation, and the fact that the $(r+4)$-th derivative of $\boldsymbol{B}$ with respect to the new parameter is bounded independently of h. Let us verify the relations (30) and (31). Recall first

$$t^q = \sum_{j=0}^{r+1} t_j^q \mathcal{L}_j(t), \quad q = 0, 1, \ldots, r+1, \quad t^{r+2} = \omega(t) + \sum_{j=0}^{r+1} t_j^{r+2} \mathcal{L}_j(t). \tag{32}$$

The divided difference $[t_0, t_0, t_1, \ldots, t_r, t_{r+1}]$ maps polynomials of degree $\leq r+1 = d-1$ to zero, and depends continuously on its arguments if applied to a smooth function. Thus $\boldsymbol{b}$ by (15) and (23) near the limit point $\boldsymbol{t}^*$ behaves like

$$\boldsymbol{b} = \left(\mathcal{O}(h^d), \mathcal{O}(h^d), \ldots, \mathcal{O}(h^d), \chi_d\, h^d + \mathcal{O}(h^{d+1})\right)^T,$$

where $\chi_i = \prod_{q=0}^{i-1} \kappa_{q,0} > 0$. On the other hand, (29) and (32) imply that

$$\begin{aligned}
\sum_{j=0}^{r+1} \boldsymbol{T}_j \mathcal{L}_j(t) &= \sum_{j=0}^{r+1} \boldsymbol{f}(\eta_j h) \mathcal{L}_j(t) \\
&= \left(\chi_1\, h\, t, \ldots, \chi_{d-1}\, h^{d-1}\, t^{d-1}, \chi_d\, h^d\, (t^d - \omega(t))\right)^T (1 + \mathcal{O}(h)).
\end{aligned}$$

But

$$\boldsymbol{B}^{(q)}(t) = \boldsymbol{b}\,\omega(t)^{(q)} + \sum_{j=0}^{r+1} \boldsymbol{T}_j \mathcal{L}_j(t)^{(q)}, \quad q = 1, 2, \ldots, r+2,$$

and (31) follows. The proof is complete. □

There is no hope that this approach could be used for all k. In fact, it fails already for $k = 1$, as we will show now. By (13), the equation (24) is replaced by

$$[t_0, t_0, t_1, \ldots, t_r, t_{r+1}]D^{-1}\boldsymbol{B} = [t_0, t_0, t_1, \ldots, t_r, t_{r+1}]\widetilde{\boldsymbol{B}} + \mathcal{O}(h),$$
$$[t_0, t_1, \ldots, t_r, t_{r+1}, t_{r+1}]D^{-1}\boldsymbol{B} = [t_0, t_1, \ldots, t_r, t_{r+1}, t_{r+1}]\widetilde{\boldsymbol{B}} + \mathcal{O}(h).$$

Further, as in the proof of Theorem 2, the first column of the Jacobian is determined from

$$\frac{\partial}{\partial \alpha_0}[t_0, t_0, t_1, \ldots, t_r, t_{r+1}]\big(\widetilde{\boldsymbol{B}} - \widetilde{\boldsymbol{f}}\big)\big|_{\boldsymbol{t}^*} = \frac{1}{\widetilde{\omega}'(\eta_0)}\widetilde{\boldsymbol{f}}'(\eta_0),$$
$$\frac{\partial}{\partial \alpha_0}[t_0, t_1, \ldots, t_r, t_{r+1}, t_{r+1}]\big(\widetilde{\boldsymbol{B}} - \widetilde{\boldsymbol{f}}\big)\big|_{\boldsymbol{t}^*} = \boldsymbol{0},$$

the last column from

$$\frac{\partial}{\partial \alpha_{r+1}}[t_0, t_0, t_1, \ldots, t_r, t_{r+1}]\big(\widetilde{\boldsymbol{B}} - \widetilde{\boldsymbol{f}}\big)\big|_{\boldsymbol{t}^*} = \boldsymbol{0},$$
$$\frac{\partial}{\partial \alpha_{r+1}}[t_0, t_1, \ldots, t_r, t_{r+1}, t_{r+1}]\big(\widetilde{\boldsymbol{B}} - \widetilde{\boldsymbol{f}}\big)\big|_{\boldsymbol{t}^*} = \frac{1}{\widetilde{\omega}'(\eta_{r+1})}\widetilde{\boldsymbol{f}}'(\eta_{r+1}),$$

and the other columns from

$$\left(\frac{\partial}{\partial t_j}[t_0, t_0, t_1, \ldots, t_r, t_{r+1}]\right)\widetilde{\boldsymbol{f}}\big|_{\boldsymbol{t}^*} = -\frac{1}{(\eta_j - \eta_0)\widetilde{\omega}'(t_j)}\widetilde{\boldsymbol{f}}'(\eta_j),$$
$$\left(\frac{\partial}{\partial t_j}[t_0, t_1, \ldots, t_r, t_{r+1}, t_{r+1}]\right)\widetilde{\boldsymbol{f}}\big|_{\boldsymbol{t}^*} = -\frac{1}{(\eta_j - \eta_{r+1})\widetilde{\omega}'(t_j)}\widetilde{\boldsymbol{f}}'(\eta_j).$$

After normalizing the Jacobian from the left by D_1^{-1}, and by D_2^{-1} from the right one obtains the matrix $A := (a_{ij})_{i,j=1}^{2d}$ with

$$a_{i,1} = \delta_{i,1},\ \ i = 1, 2, \ldots, 2d,$$
$$a_{i,2d} = 0,\ \ a_{i+d,2d} = 1,\ \ i = 1, 2, \ldots, d,$$

and

$$a_{i,j} = \eta_{j-1}^i - \eta_{j-1}^{i-1},\ \ a_{i+d,j} = \eta_{j-1}^i,\ \ \ i = 1, 2, \ldots, d,\ \ j = 2, 3, \ldots, 2d-1.$$

A simple rank preserving transformation

$$a_{i,j} \mapsto a_{i,j} - a_{i-1,j},\ \ i = 2d, 2d-1, \ldots, d+1,\ \ j = 1, 2, \ldots, 2d,$$

transforms A to a matrix with row i equal to row $i+d$ for $i = 2, 3, \ldots, d$. It is now easy to see that the rank of the matrix A is $d+1$, and consequently $\dim\ker A = d-1$. Thus, since the Jacobian is singular, some other approach such as [1], pp. 154–155, should be applied to carry out the asymptotic analysis.

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