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Lattices on tetrahedral partitions

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Abstract In this paper, four-pencil lattices on tetrahedral partitions are studied. The explicit representation of a lattice, based upon barycentric coordinates, enables us to extend the lattice from a single tetrahedron to a tetrahedral partition. It is shown that the number of degrees of freedom is equal to the number of vertices of the tetrahedral partition. The proof is based on a lattice split approach.

Keywords Lattice · Tetrahedron · Interpolation

Mathematics Subject Classification (2000) 41A05 · 41A63 · 65D05

1 Introduction

The existence and the uniqueness of the solution of a multivariate polynomial interpolation problem essentially depends on the geometrical distribution of interpolation points in a domain. It is well-known that the Lagrange interpolation problem for a set of $\binom{n+d}{d}$ interpolation points is correct in Π_n^d (the space of polynomials in d variables of total degree $\leq n$) if and only if the points do not lie on an algebraic hypersurface of degree $\leq n$. This property is hard to verify on the run, particularly by numerical computations.

There have been various attempts to construct a lattice on $\binom{n+d}{d}$ interpolation points that admits a correct interpolation problem in Π_n^d in advance. Since ([3]), these constructions mainly consist of choosing the points as appropriate intersections of hyperplanes. It is well-known that the lattices

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which satisfy the GC (geometric characterization) condition admit correct interpolation ([2],[3]). Among them, principal lattices and the more general $(d+1)$-pencil lattices ([7]) are perhaps the most important.

In [5] a new approach to $(d+1)$-pencil lattice, based upon control points, has been presented and the barycentric coordinates of lattice points were derived. In this paper we show that such a representation enables a natural extension of a four-pencil lattice from a tetrahedron to a regular tetrahedral partition. Lattices on regular tetrahedral partitions, where the points on adjacent triangles coincide, are important, since they provide at least continuous piecewise polynomial interpolants. In [4] this problem has already been discussed for the case $d = 2$, i.e., three-pencil lattices on triangulations. It turns out that the $3D$ case is much more complicated.

The paper is organized as follows. In Section 2 some conditions, which allow the construction of lattices on adjacent tetrahedrons are given. In Section 3 the extension of a four-pencil lattice from a tetrahedron to a tetrahedral partition is presented. The extended lattice has V degrees of freedom, where V is the number of vertices of a tetrahedral partition.

2 Matching of two lattices

For our purpose the ordering of the vertices of a tetrahedron will be important. Therefore

$$\triangle := \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle, \qquad \boldsymbol{T}_i \in \mathbf{R}^3,\ i = 0,1,2,3,$$

will denote a tetrahedron with a prescribed order of vertices $\boldsymbol{T}_i$. A four-pencil lattice of order n on $\triangle$ is generated by particular four pencils of $n+1$ planes, where each pencil intersects at a center $\boldsymbol{C}_i$, $i = 0,1,2,3$, a line in $\mathbf{R}^3$. The lattice is actually based upon four control points $\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3$, where $\boldsymbol{P}_i \in \mathbf{R}^3$ lies on the line through the vertices $\boldsymbol{T}_i$ and $\boldsymbol{T}_{i+1}$ outside of the segment $\boldsymbol{T}_i\boldsymbol{T}_{i+1}$. The center $\boldsymbol{C}_i$ is then a line through the control points $\boldsymbol{P}_i$ and $\boldsymbol{P}_{i+1}$ (Fig. 1). Note that here and throughout the paper, the indices of control points, vertices, centers and lattice parameters are assumed to be taken modulo $d+1$.

Let $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$, $\gamma_i \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}$, denote an index vector and let $|\boldsymbol{\gamma}| := \sum_{i=0}^{3} \gamma_i$. In [5], the barycentric coordinates of a four-pencil lattice on a tetrahedron $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle$ w.r.t. $\triangle$ were determined by four free parameters $\boldsymbol{\xi} = (\xi_0, \xi_1, \xi_2, \xi_3)$, $\xi_i > 0$, as

$$\begin{aligned} B_{\boldsymbol{\gamma}}(\boldsymbol{\xi}) &= \frac{1}{D}\left(\alpha^{n-\gamma_0}\left[\gamma_0\right]_\alpha, \xi_0\alpha^{n-\gamma_0-\gamma_1}\left[\gamma_1\right]_\alpha, \xi_0\xi_1\alpha^{\gamma_3}\left[\gamma_2\right]_\alpha, \xi_0\xi_1\xi_2\left[\gamma_3\right]_\alpha\right), \\ D &= \alpha^{n-\gamma_0}\left[\gamma_0\right]_\alpha + \xi_0\alpha^{n-\gamma_0-\gamma_1}\left[\gamma_1\right]_\alpha + \xi_0\xi_1\alpha^{\gamma_3}\left[\gamma_2\right]_\alpha + \xi_0\xi_1\xi_2\left[\gamma_3\right]_\alpha, \end{aligned} \tag{1}$$

where $\boldsymbol{\gamma} \in \mathbf{N}_0^4$, $|\boldsymbol{\gamma}| = n$,

$$\alpha := \sqrt[n]{\prod_{i=0}^{3} \xi_i} > 0, \quad \text{and} \quad [j]_\alpha := \sum_{i=0}^{j-1} \alpha^i = \begin{cases} j, & \alpha = 1, \\ \dfrac{1-\alpha^j}{1-\alpha}, & \alpha \neq 1, \end{cases} \quad j \in \mathbf{N}_0.$$

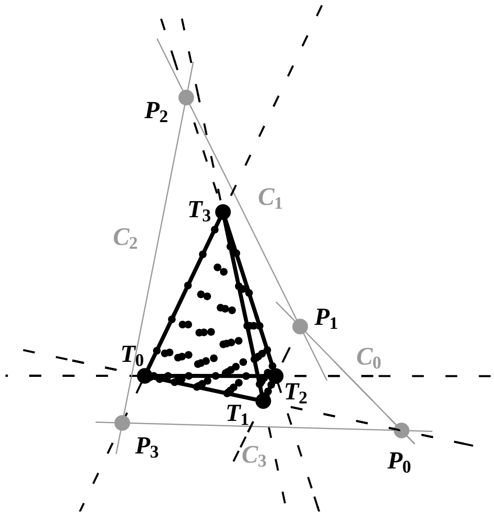

Fig. 1 A four-pencil lattice with its control points $\boldsymbol{P}_i$ and centers $\boldsymbol{C}_i$.

The results which follow will be a basis for the extension of a four-pencil lattice of order n from a tetrahedron to a tetrahedral partition. We will first answer the question when two lattices on tetrahedrons $\triangle$ and $\triangle'$ match on a common face.

Theorem 1 *Let* $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle$ *and* $\triangle' = \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \boldsymbol{T}'_2, \boldsymbol{T}'_3 \rangle$ *be given tetrahedrons and let* $B_{\boldsymbol{\gamma}}(\boldsymbol{\xi})$ *and* $B_{\boldsymbol{\gamma}}(\boldsymbol{\xi}')$ *be the barycentric coordinates w.r.t.* $\triangle$ *and* $\triangle'$ *of four-pencil lattices of order* n *on* $\triangle$ *and* $\triangle'$*, respectively. Let* $\alpha^n = \prod_{j=0}^{3} \xi_j$ *and* $\alpha'^n = \prod_{j=0}^{3} \xi'_j$*. The lattices coincide on the common edge*

$$\langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1} \rangle = \langle \boldsymbol{T}'_{i'_0}, \boldsymbol{T}'_{i'_1} \rangle, \qquad 0 \le i_0 < i_1 \le 3, \quad 0 \le i'_0 < i'_1 \le 3,$$

iff

$$\left(\prod_{j=i_0}^{i_1-1} \xi_j \right) \alpha' = \left(\prod_{j=i'_0}^{i'_1-1} \xi'_j \right) \alpha, \tag{2}$$

in the case $n = 2$*, and*

$$\left(\prod_{j=i_0}^{i_1-1} \xi_j = \prod_{j=i'_0}^{i'_1-1} \xi'_j \quad \text{and} \quad \alpha' = \alpha \right) \quad \text{or} \quad \left(\prod_{j=i_0}^{i_1-1} \xi_j = \alpha^n \prod_{j=i'_0}^{i'_1-1} \xi'_j \quad \text{and} \quad \alpha' \alpha = 1 \right), \tag{3}$$

for $n \ge 3$.

Proof By (1), the barycentric coordinates w.r.t. $\langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1} \rangle$, $0 \le i_0 < i_1 \le 3$, of the first lattice on $\langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1} \rangle$ are

$$\left(\frac{[n]_\alpha - [\ell]_\alpha}{[n]_\alpha - [\ell]_\alpha + [\ell]_\alpha \prod_{j=i_0}^{i_1-1} \xi_j}, \frac{[\ell]_\alpha \prod_{j=i_0}^{i_1-1} \xi_j}{[n]_\alpha - [\ell]_\alpha + [\ell]_\alpha \prod_{j=i_0}^{i_1-1} \xi_j} \right), \quad \ell = 0, \ldots, n.$$

Therefore the lattices coincide on $\langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1} \rangle = \langle \boldsymbol{T}'_{i'_0}, \boldsymbol{T}'_{i'_1} \rangle$, $0 \le i_0 < i_1 \le 3$, $0 \le i'_0 < i'_1 \le 3$, iff

$$\left(\frac{[\ell]_\alpha}{[n]_\alpha - [\ell]_\alpha}\right) \prod_{j=i_0}^{i_1-1} \xi_j = \left(\frac{[\ell]_{\alpha'}}{[n]_{\alpha'} - [\ell]_{\alpha'}}\right) \prod_{j=i'_0}^{i'_1-1} \xi'_j, \quad \ell = 1, 2, \ldots, n-1. \tag{4}$$

Since the case $n = 2$ is straightforward, let $n \ge 3$. By defining $\zeta_0 := \prod_{j=i_0}^{i_1-1} \xi_j$ and $\zeta'_0 := \prod_{j=i'_0}^{i'_1-1} \xi'_j$, the proof of [4, Theorem 2] shows that the system (4) has precisely two solutions and they are given by (3). $\square$

Since dealing with barycentric coordinates one can similarly show that the lattices coincide on the common edge

$$\langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1} \rangle = \langle \boldsymbol{T}'_{i'_0}, \boldsymbol{T}'_{i'_1} \rangle, \qquad 0 \le i_0 < i_1 \le 3, \quad 0 \le i'_1 < i'_0 \le 3,$$

iff

$$\prod_{j=i_0}^{i_1-1} \xi_j = \alpha' \alpha \prod_{j=i'_1}^{i'_0-1} \xi'^{-1}_j, \tag{5}$$

for the case $n = 2$, and

$$\left(\prod_{j=i_0}^{i_1-1} \xi_j = \alpha^n \prod_{j=i'_1}^{i'_0-1} \xi'^{-1}_j \text{ and } \alpha' = \alpha\right) \text{ or } \left(\prod_{j=i_0}^{i_1-1} \xi_j = \prod_{j=i'_1}^{i'_0-1} \xi'^{-1}_j \text{ and } \alpha'\alpha = 1\right), \tag{6}$$

for $n \ge 3$.

Consider now two four-pencil lattices that share a lattice on a common triangle of tetrahedrons $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle$ and $\triangle' = \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \boldsymbol{T}'_2, \boldsymbol{T}'_3 \rangle$ (Fig. 2).

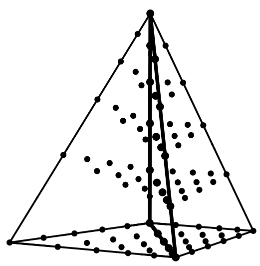

Fig. 2 Matching of two lattices on a common triangle of tetrahedrons.

Corollary 1 *Let* $\alpha^n = \prod_{j=0}^{3} \xi_j \ne 1$ *and let*

$$\widetilde{\triangle} := \langle \boldsymbol{T}_{i_0}, \boldsymbol{T}_{i_1}, \boldsymbol{T}_{i_2} \rangle = \langle \boldsymbol{T}'_{i'_0}, \boldsymbol{T}'_{i'_1}, \boldsymbol{T}'_{i'_2} \rangle,$$

$0 \le i_0 < i_1 < i_2 \le 3$, $0 \le i'_0 < i'_1 < i'_2 \le 3$, *be the common triangle of tetrahedrons. Then the lattices coincide on* $\widetilde{\triangle}$ *iff*

$$\prod_{j=i_k}^{i_{k+1}-1} \xi_j = \prod_{j=i'_k}^{i'_{k+1}-1} \xi'_j, \quad k = 0, 1, \quad \textit{and} \quad \alpha' = \alpha. \tag{7}$$

Proof The lattices coincide on $\widetilde{\triangle}$ iff they coincide on all three edges of $\widetilde{\triangle}$. For $n = 2$, the lattices coincide on $\widetilde{\triangle}$ iff the equations

$$\left(\prod_{j=i_k}^{i_{k+\ell}-1} \xi_j \right) \alpha' = \left(\prod_{j=i'_k}^{i'_{k+\ell}-1} \xi'_j \right) \alpha, \quad k = 0, 1, \;\; \ell = 1, \ldots, 2-k,$$

that are equivalent to (7), hold. Let now $n \ge 3$ and recall (3). Then for the first possibility, $\alpha' = \alpha$, the lattices coincide on $\widetilde{\triangle}$ iff (7) holds, and for the second one, $\alpha'\alpha = 1$, iff

$$\prod_{j=i_k}^{i_{k+\ell}-1} \xi_j = \alpha^n \prod_{j=i'_k}^{i'_{k+\ell}-1} \xi'_j, \qquad k = 0, 1, \;\; \ell = 1, \ldots, 2-k. \tag{8}$$

But from (8) we obtain $\alpha = \alpha^2$, which is a contradiction, since $\alpha \ne 1$. $\square$

Note that with the assumption $\alpha = 1$ some further analysis could be easier but we would loose a degree of freedom ([4]).

The following corollary will be important when dealing with tetrahedral partitions in the next section.

Corollary 2 *Let* $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle$ *be a tetrahedron and let* $B_{\boldsymbol{\gamma}}(\boldsymbol{\xi})$ *be the barycentric coordinates w.r.t.* $\triangle$ *of the lattice on* $\triangle$*. Let* $B_{\boldsymbol{\gamma}}(\boldsymbol{\xi}')$ *be the barycentric coordinates w.r.t.* $\triangle'$ *of the lattice on* $\triangle'$*, where*

$$\triangle' = \langle \boldsymbol{T}'_0, \boldsymbol{T}'_1, \boldsymbol{T}'_2, \boldsymbol{T}'_3 \rangle := \langle \boldsymbol{T}_{\sigma(0)}, \boldsymbol{T}_{\sigma(1)}, \boldsymbol{T}_{\sigma(2)}, \boldsymbol{T}_{\sigma(3)} \rangle, \quad \sigma \in C_4,$$

and C_4 *is the cyclic group of order* 4*. Let* $\alpha^n = \prod_{j=0}^{3} \xi_j \ne 1$*. Then the lattices coincide on* $\triangle$ *iff*

$$\xi'_i = \xi_{\sigma(i)}, \quad i = 0, 1, 2, 3. \tag{9}$$

Proof The lattices coincide on $\triangle$ iff they coincide on all edges of $\triangle$. Using relations (2), (3), (5) and (6), Corollary 1, and the fact $|\sigma(i) - \sigma(j)| = |i - j|$, lattices match on the edges $\langle \boldsymbol{T}'_j, \boldsymbol{T}'_{j+1} \rangle = \langle \boldsymbol{T}_{\sigma(j)}, \boldsymbol{T}_{\sigma(j+1)} \rangle$, $j = 0, 1, 2$, and $\langle \boldsymbol{T}'_0, \boldsymbol{T}'_2 \rangle = \langle \boldsymbol{T}_{\sigma(0)}, \boldsymbol{T}_{\sigma(2)} \rangle$ iff (9) holds. It is then straightforward to verify the matching of the lattices on the other two edges. $\square$

Note that for $\sigma \notin D_4$, where D_4 is the dihedral group of order 4, the positions of centers would not be the same for both lattices and therefore the lattices would not coincide. Since we will not need the whole dihedral group in the next section, only the cyclic subgroup has been considered.

3 Tetrahedral partition

Now, our goal is to extend the four-pencil lattice from a single tetrahedron to a regular tetrahedral partition. Recall that a partition is regular if every pair of adjacent tetrahedrons have either a point, an edge or the whole triangle in common, and that each tetrahedron has at least one facet in common with another tetrahedron. This extension should be done in such a way that the lattices on any two adjacent tetrahedrons coincide on common faces of tetrahedrons (Fig 2). We will not only prove that such an extension exists, but even more, that a lattice obtained in this way has the maximal possible number of parameters of freedom. First consider a particular case of tetrahedral partition, i.e., a star in $\mathbf{R}^3$. Recall that a star in $\mathbf{R}^3$ of degree m is a tetrahedral partition with exactly one inner vertex (of the degree m) ([6]).

Lemma 1 *Let $\mathcal{S}$ be a star of tetrahedrons of degree $V-1$. Then there exists a four-pencil lattice on $\mathcal{S}$ and there are V degrees of freedom to construct it.*

Proof We will first prove the lemma for the minimal star $\mathcal{S}_0$, which consists of four tetrahedrons. Then we will show how an arbitrary star can be obtained from $\mathcal{S}_0$ and how the lattice on $\mathcal{S}_0$ can be extended to the lattice on $\mathcal{S}$. With

$$\langle \boldsymbol{T}_{j_0}, \boldsymbol{T}_{j_1}, \boldsymbol{T}_{j_2} \rangle_i, \quad j_0, \ldots, j_k \in \{0,1,2,3\},$$

we will denote the facet of the i-th tetrahedron. Let $\widetilde{\mathcal{S}}$ be the triangulation obtained from a star $\mathcal{S}$ by removing the interior point of $\mathcal{S}$ and all its incident edges. Further, let the inner point of a star for all tetrahedrons in $\mathcal{S}$ be labeled by $\boldsymbol{T}_3$ and let the other vertices of tetrahedrons in $\mathcal{S}_0$ be ordered as in Fig. 3. Here and throughout the proof the most important is to assure

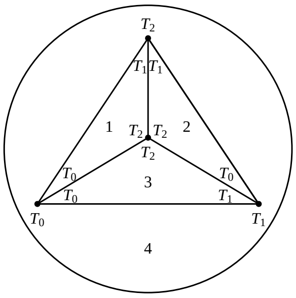

Fig. 3 On a sphere embedded triangulation $\widetilde{\mathcal{S}_0}$ obtained from a minimal star $\mathcal{S}_0$ by removing the interior point $\boldsymbol{T}_3$ and all its incident edges.

that a common triangle of any two adjacent tetrahedrons is of the form

$$\langle \boldsymbol{T}_{j_0}, \boldsymbol{T}_{j_1}, \boldsymbol{T}_{j_2} \rangle_i = \langle \boldsymbol{T}_{j'_0}, \boldsymbol{T}_{j'_1}, \boldsymbol{T}_{j'_2} \rangle_{i'}, \qquad j_0 < j_1 < j_2, \ j'_0 < j'_1 < j'_2. \quad (10)$$

Note that we could also use some other ordering of vertices in $\widetilde{\mathcal{S}_0}$, which satisfies (10) for all common triangles. Since $\mathcal{S}_0$ is a star of degree 4, we have to prove that there are 5 degrees of freedom to construct a lattice on it. Let the lattice on the i-th tetrahedron be determined by parameters $\xi_j^{(i)}$, $j = 0, 1, 2, 3$. We have to assure the matching of lattices on the following triangles (Fig. 3)

$$\begin{aligned}
\langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_1 &= \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_4, & \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_2 &= \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_4,\\
\langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_3 &= \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_3 \rangle_4, & \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_1 &= \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_2,\\
\langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_2 &= \langle \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_3, & \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_1 &= \langle \boldsymbol{T}_0, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle_3.
\end{aligned}$$

By Corollary 1, all parameters $\xi_j^{(i)}$, $i = 1, 2, 3, 4$, $j = 0, 1, 2, 3$, are determined by 5 parameters $\xi_0^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}$ and $\xi_0^{(2)}$ as

$$\begin{aligned}
\boldsymbol{\xi}^{(1)} &= \left(\xi_0^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}\right), \ \boldsymbol{\xi}^{(2)} = \left(\xi_0^{(2)}, \xi_1^{(1)}, \xi_2^{(1)}, \frac{\xi_0^{(1)}\xi_3^{(1)}}{\xi_0^{(2)}}\right),\\
\boldsymbol{\xi}^{(3)} &= \left(\frac{\xi_0^{(1)}}{\xi_0^{(2)}}, \xi_1^{(1)}\xi_0^{(2)}, \xi_2^{(1)}, \xi_3^{(1)}\right), \ \boldsymbol{\xi}^{(4)} = \left(\frac{\xi_0^{(1)}}{\xi_0^{(2)}}, \xi_0^{(2)}, \xi_1^{(1)}\xi_2^{(1)}, \xi_3^{(1)}\right). \quad (11)
\end{aligned}$$

Let E denote the number of edges, F the number of triangles, and V_k the number of vertices with degree k in $\widetilde{\mathcal{S}}$. Since $2E = \sum_k kV_k$ and $3F = 2E$, the Euler formula implies $\sum_k V_k(6-k) = 12$. Therefore, $\widetilde{\mathcal{S}}$ must have a vertex of degree less than six. Because every edge of $\widetilde{\mathcal{S}}$ must lie on two distinct triangles, each vertex has degree greater than two. Thus there is at least one vertex with degree 3, 4 or 5 in $\widetilde{\mathcal{S}}$. Let now $\mathcal{S}'$ denote a star of degree $V-2$. Any new star $\mathcal{S}$ of degree $V-1$ can be obtained from $\mathcal{S}'$ by one of the following operations (see [1], e.g.). Add a new vertex into $\widetilde{\mathcal{S}'}$ to split

(a) one tetrahedron into three tetrahedrons (Fig. 4),
(b) two tetrahedrons into four tetrahedrons (Fig. 5),
(c) three tetrahedrons into five tetrahedrons (Fig. 6).

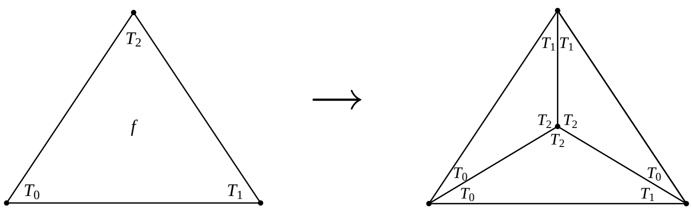

Fig. 4 Adding a new vertex in order to split one tetrahedron into three tetrahedrons (Clough-Tocher split).

With all these operations we add one new vertex to $\mathcal{S}'$, so we have to prove that for each operation the number of free parameters increases by one. The relations that determine the parameters of new tetrahedrons after the operation (a) (Fig. 4) are similar to the relations in (11) and thus this operation brings one new parameter up. Let us now prove the same for the operation

(b). Without loss of generality we can assume that the faces $f_1, f_2 \in \widetilde{\mathcal{S}}'$ of two selected tetrahedrons as also the newly obtained tetrahedrons are ordered as in Fig. 5. After the split we have to assure the matching on triangles

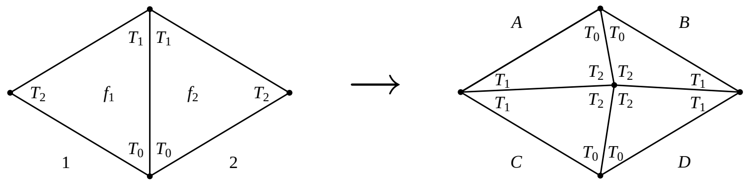

Fig. 5 A new vertex splits two tetrahedrons into four tetrahedrons.

$$\begin{aligned}
&\langle \boldsymbol{T}_1,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_1 = \langle \boldsymbol{T}_0,\boldsymbol{T}_1,\boldsymbol{T}_3\rangle_A, \quad \langle \boldsymbol{T}_0,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_1 = \langle \boldsymbol{T}_0,\boldsymbol{T}_1,\boldsymbol{T}_3\rangle_C,\\
&\langle \boldsymbol{T}_1,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_2 = \langle \boldsymbol{T}_0,\boldsymbol{T}_1,\boldsymbol{T}_3\rangle_B, \quad \langle \boldsymbol{T}_0,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_2 = \langle \boldsymbol{T}_0,\boldsymbol{T}_1,\boldsymbol{T}_3\rangle_D,\\
&\langle \boldsymbol{T}_0,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_A = \langle \boldsymbol{T}_0,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_B, \quad \langle \boldsymbol{T}_1,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_A = \langle \boldsymbol{T}_1,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_C,\\
&\langle \boldsymbol{T}_1,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_B = \langle \boldsymbol{T}_1,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_D, \quad \langle \boldsymbol{T}_0,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_C = \langle \boldsymbol{T}_0,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_D.
\end{aligned}$$

Using Corollary 1, the number of degrees of freedom increases again by one,

$$\begin{aligned}
\boldsymbol{\xi}^{(1)} &= \left(\xi_0^{(1)},\xi_1^{(1)},\xi_2^{(1)},\xi_3^{(1)}\right),\ \boldsymbol{\xi}^{(2)} = \left(\xi_0^{(1)},\xi_1^{(2)},\frac{\xi_1^{(1)}\xi_2^{(1)}}{\xi_1^{(2)}},\xi_3^{(1)}\right),\\
\boldsymbol{\xi}^{(A)} &= \left(\xi_1^{(1)},\xi_1^{(A)},\frac{\xi_2^{(1)}}{\xi_1^{(A)}},\xi_0^{(1)}\xi_3^{(1)}\right),\ \boldsymbol{\xi}^{(B)} = \left(\xi_1^{(2)},\frac{\xi_1^{(1)}\xi_1^{(A)}}{\xi_1^{(2)}},\frac{\xi_2^{(1)}}{\xi_1^{(A)}},\xi_0^{(1)}\xi_3^{(1)}\right),\\
\boldsymbol{\xi}^{(C)} &= \left(\xi_0^{(1)}\xi_1^{(1)},\xi_1^{(A)},\frac{\xi_2^{(1)}}{\xi_1^{(A)}},\xi_3^{(1)}\right),\ \boldsymbol{\xi}^{(D)} = \left(\xi_0^{(1)}\xi_1^{(2)},\frac{\xi_1^{(1)}\xi_1^{(A)}}{\xi_1^{(2)}},\frac{\xi_2^{(1)}}{\xi_1^{(A)}},\xi_3^{(1)}\right).
\end{aligned}$$

The operation (c) splits three tetrahedrons into five tetrahedrons (Fig. 6). Again without loss of generality we can assume that the triangles f_1, f_2 and f_3 as also the newly obtained tetrahedrons are ordered as in Fig. 6. We have now 10 common triangles where the matching has to be assured,

$$\begin{aligned}
&\langle \boldsymbol{T}_0,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_1 = \langle \boldsymbol{T}_0,\boldsymbol{T}_1,\boldsymbol{T}_3\rangle_A, \quad \langle \boldsymbol{T}_1,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_1 = \langle \boldsymbol{T}_0,\boldsymbol{T}_1,\boldsymbol{T}_3\rangle_E,\\
&\langle \boldsymbol{T}_1,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_2 = \langle \boldsymbol{T}_0,\boldsymbol{T}_1,\boldsymbol{T}_3\rangle_D, \quad \langle \boldsymbol{T}_0,\boldsymbol{T}_1,\boldsymbol{T}_3\rangle_3 = \langle \boldsymbol{T}_0,\boldsymbol{T}_1,\boldsymbol{T}_3\rangle_B,\\
&\langle \boldsymbol{T}_1,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_3 = \langle \boldsymbol{T}_0,\boldsymbol{T}_1,\boldsymbol{T}_3\rangle_C, \quad \langle \boldsymbol{T}_0,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_A = \langle \boldsymbol{T}_0,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_B,\\
&\langle \boldsymbol{T}_1,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_B = \langle \boldsymbol{T}_0,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_C, \quad \langle \boldsymbol{T}_1,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_C = \langle \boldsymbol{T}_1,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_D,\\
&\langle \boldsymbol{T}_0,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_D = \langle \boldsymbol{T}_0,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_E, \quad \langle \boldsymbol{T}_1,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_A = \langle \boldsymbol{T}_1,\boldsymbol{T}_2,\boldsymbol{T}_3\rangle_E,
\end{aligned}$$

and again Corollary 1 proves the desired

$$\begin{aligned}
\boldsymbol{\xi}^{(1)} &= \left(\xi_0^{(1)},\xi_1^{(1)},\xi_2^{(1)},\xi_3^{(1)}\right),\ \boldsymbol{\xi}^{(B)} = \left(\xi_0^{(3)},\frac{\xi_0^{(1)}\xi_1^{(1)}\xi_1^{(A)}}{\xi_0^{(3)}},\frac{\xi_2^{(1)}}{\xi_1^{(A)}},\xi_3^{(1)}\right),\\
\boldsymbol{\xi}^{(2)} &= \left(\xi_0^{(1)},\xi_1^{(2)},\frac{\xi_1^{(1)}\xi_2^{(1)}}{\xi_1^{(2)}},\xi_3^{(1)}\right),\boldsymbol{\xi}^{(C)} = \left(\frac{\xi_0^{(1)}\xi_1^{(2)}}{\xi_0^{(3)}},\frac{\xi_1^{(1)}\xi_1^{(A)}}{\xi_1^{(2)}},\frac{\xi_2^{(1)}}{\xi_1^{(A)}},\xi_0^{(3)}\xi_3^{(1)}\right),
\end{aligned}$$

$$\boldsymbol{\xi}^{(3)} = \left(\xi_0^{(3)}, \frac{\xi_0^{(1)}\xi_1^{(2)}}{\xi_0^{(3)}}, \frac{\xi_1^{(1)}\xi_2^{(1)}}{\xi_1^{(2)}}, \xi_3^{(1)}\right), \boldsymbol{\xi}^{(D)} = \left(\xi_1^{(2)}, \frac{\xi_1^{(1)}\xi_1^{(A)}}{\xi_1^{(2)}}, \frac{\xi_2^{(1)}}{\xi_1^{(A)}}, \xi_0^{(1)}\xi_3^{(1)}\right),$$

$$\boldsymbol{\xi}^{(A)} = \left(\xi_0^{(1)}\xi_1^{(1)}, \xi_1^{(A)}, \frac{\xi_2^{(1)}}{\xi_1^{(A)}}, \xi_3^{(1)}\right), \boldsymbol{\xi}^{(E)} = \left(\xi_1^{(1)}, \xi_1^{(A)}, \frac{\xi_2^{(1)}}{\xi_1^{(A)}}, \xi_0^{(1)}\xi_3^{(1)}\right).$$

Similarly one can verify that all other orderings of the triangles f_1, f_2 and f_3

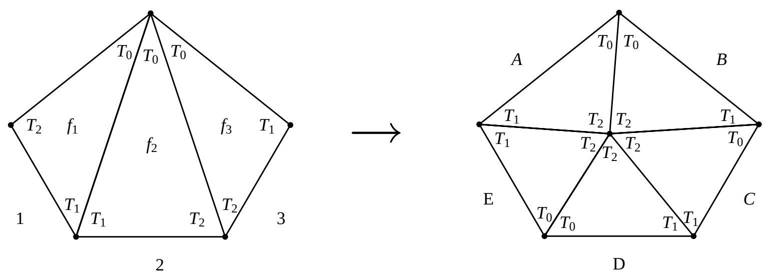

Fig. 6 Three tetrahedrons are replaced with five tetrahedrons.

before the operations (b) and (c) as also the orderings of the newly obtained tetrahedrons after these operations provide the same results as soon as these orderings are such that (10) holds for all common triangles. This follows from the fact that if the latter holds, the lattice on a particular tetrahedron or triangle is uniquely determined, as soon as it is known on two triangular faces or edges, respectively. Moreover, there is always possible to order the vertices of the newly obtained tetrahedrons such that (10) holds for all common triangles. Indeed, label the newly added vertex by $\boldsymbol{T}_2$ and the interior vertex of the star by $\boldsymbol{T}_3$ for all new tetrahedrons. Further, label the remaining two vertices of each of these tetrahedrons by $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$ in the same order as were on these edges labeled tetrahedrons which were split. □

We are now able to generalize [4, Theorem 3] from triangulations to the next important case, i.e., tetrahedral partitions.

Theorem 2 *Let $\mathcal{T}$ be a regular tetrahedral partition with V vertices. Then there exists a four-pencil lattice on $\mathcal{T}$ which is determined by V parameters.*

Proof By Corollary 1 the theorem obviously holds for two tetrahedrons and by Lemma 1 for a star. Suppose now that the lattice exists on a subpartition $\mathcal{T}'$ of the tetrahedral partition $\mathcal{T}$ and is determined by V' parameters, where V' is the number of vertices of $\mathcal{T}'$. Now take a vertex $\boldsymbol{T}$ at the boundary of $\mathcal{T}'$. Our goal is to prove the existence of the lattice on $\mathcal{T}' \cup \mathcal{S}$, where $\mathcal{S}$ is a star (interior or boundary in $\mathcal{T}$) around the vertex $\boldsymbol{T}$. We have to use the procedure of Lemma 1 for $\mathcal{S}$ in such a way that the lattices on $\mathcal{T}'$ and $\mathcal{S}$ will match on $\mathcal{S}' := \mathcal{S} \cap \mathcal{T}' \neq \emptyset$. Let the inner point of $\mathcal{S}$ be labeled by $\boldsymbol{T}_3$ for all tetrahedrons in $\mathcal{S}$. Since the tetrahedrons in $\mathcal{S}'$ already have prescribed order of vertices, we have to use Corollary 2 to reorder these vertices such that the inner point of $\mathcal{S}$ becomes $\boldsymbol{T}_3$ also for all tetrahedrons in $\mathcal{S}'$. This does not change the number of free parameters. Furthermore, Lemma 1 shows that in order to prove the existence of the lattice on $\mathcal{T}' \cup \mathcal{S}$, we only have to find such an ordering of the vertices of tetrahedrons in $\mathcal{S}\backslash\mathcal{S}'$, that assures (10) for all common triangles in $\mathcal{S}$ (the relation (10) already holds for all common triangles in $\mathcal{S}'$). Note that if such an ordering exists, then it can obviously be constructed by the procedure described in Lemma 1. Let us describe one of the possible orderings of vertices of tetrahedrons in $\mathcal{S}\backslash\mathcal{S}'$, which satisfies the given requirements. Recall that the inner vertex of $\mathcal{S}$ should be labeled by $\boldsymbol{T}_3$ for all tetrahedrons in $\mathcal{S}\backslash\mathcal{S}'$. Denote the part of $\mathcal{S}$, where the orderings of the tetrahedrons are already determined, by $\widetilde{\mathcal{S}}$. Consider now that $\mathcal{S}$ is obtained from $\mathcal{S}'$ by adding a tetrahedron at a time, such that the newly added tetrahedron $\triangle$ has always at least one and at most three common triangles with $\widetilde{\mathcal{S}}$. This can be assured, since $\mathcal{T}$ is regular. The new tetrahedron can have a vertex which is not in $\widetilde{\mathcal{S}}$. In this case, this vertex should be labeled by $\boldsymbol{T}_2$. Since the inner vertex of the $\mathcal{S}$ is labeled by $\boldsymbol{T}_3$ for all tetrahedrons in $\mathcal{S}$, the remaining two vertices of the tetrahedron have to be labeled by $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$ in such a way that (10) holds for a common triangle. It is obvious that one of both possibilities is appropriate. If, on the other hand, all vertices of $\triangle$ are in $\widetilde{\mathcal{S}}$, then the vertex which has been added as the last one has to be labeled by $\boldsymbol{T}_2$. Since the label $\boldsymbol{T}_3$ is reserved for the inner vertex of $\mathcal{S}$, the remaining two vertices should be labeled by $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$ such that again (10) holds for all (two or three) common triangles. This is assured if the vertex that was added the last but one is labeled by $\boldsymbol{T}_1$. Thus we have ordered the vertices of all tetrahedrons in $\mathcal{S}$ in such a way that (10) is assured for all common triangles. Using Corollary 1 and the fact that the lattice on a particular tetrahedron or triangle is uniquely determined if it is already determined on at least two facets, we have proved the existence of the lattice on $\mathcal{T}' \cup \mathcal{S}$. Since $\mathcal{T}' \cap \mathcal{S} = \mathcal{S}'$, the number of parameters that describe the lattice increases exactly by the number of vertices added to the subpartition $\mathcal{T}'$. Indeed, for each vertex $\boldsymbol{T}$, $\boldsymbol{T} \notin \mathcal{S}'$, Corollary 1 brings one new parameter up. By continuing this process we finally prove the existence of the lattice on $\mathcal{T}$, which is determined by V parameters. $\square$

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